Supersaturation and the Container Method

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Container Theorem

Definition

- For $S \subset V(\mathcal{H})$ the **degree of** S is $d(S) \; := \; |\{F \in \mathcal{H} : \; S \subset F\}|$.
- $\Delta_\ell(\mathcal{H}) := \; \max \, \{d(S) : \; S \subset V(\mathcal{H}), \; |S| = \ell\}$.

Balogh–Morris–Samotij [2015], Saxton–Thomason [2015]

Fix $k \in \mathbb{Z}^+$ and $p \in [0,1]$. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for every $\ell \in [k]$,

$$\Delta_\ell(\mathcal{H}) \leqslant p^{\ell - 1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

Then there is a family $\mathcal{C}$ of subsets of $V(\mathcal{H})$ such that

- for every independent set I there is a $C \in \mathcal{C}$ such that $I \subset C$,
- $|\mathcal{C}| \leqslant 2^{p \cdot |V| \cdot \log |V|}$,
- for every $C \in \mathcal{C}$: $e(\mathcal{H}[C]) = o(e(\mathcal{H}))$.

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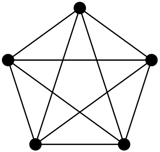

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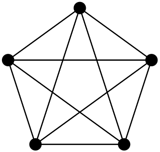

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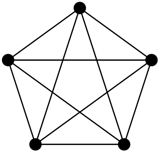

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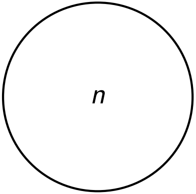

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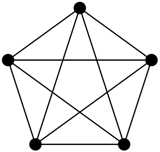

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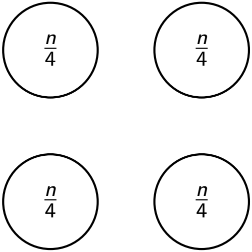

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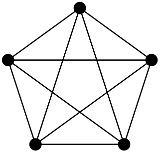

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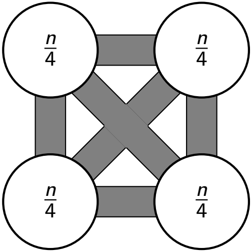

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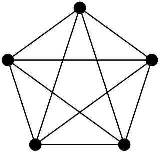

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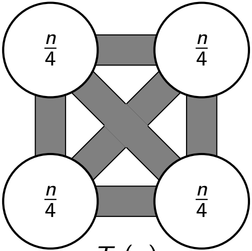

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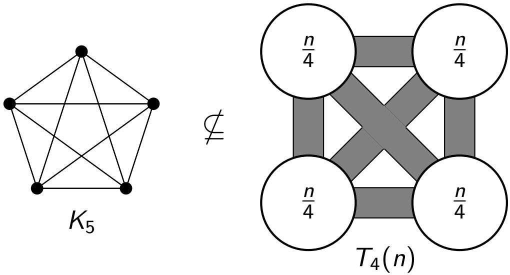

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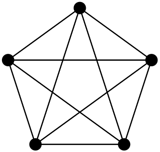

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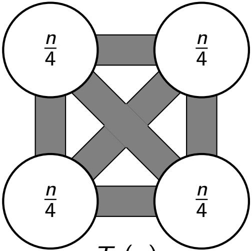

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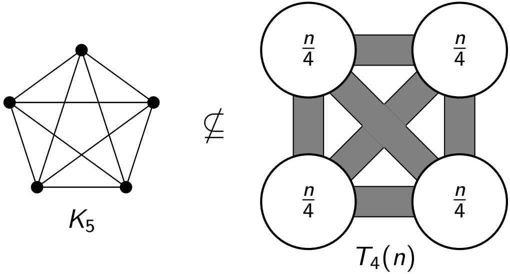

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For every $k \geq 3$:

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For every $k \geq 3$ and $\varepsilon > 0$ there is an $\alpha > 0$ that for n large if

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There are $2^{(1+o(1))\cdot \mathrm{ex}(n,K_k)}$ K_k-free graphs on n vertices.

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Theorem (Kolaitis–Prömel–Rothschild [1987])

For every fixed k almost all K_k-free graphs are $(k-1)$-partite.

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- "almost all type of results": stability method + machinery of Prömel- Steger; Balogh- Bollobás- Simonovits.

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  $|A| \geqslant (1 + o(1)) \cdot \mathrm{ex}(n, k)$ or if $|A| \geqslant 1000 \cdot \mathrm{ex}(n, k)$?
- How smooth is $\mathrm{ex}(n, k)$ for fixed k?

Linear hyper cycles

Definition

CORE vertices of a 5-cycle: $\{1, 2, 3, 4, 5\}$

GRAPH CYCLE C_5:

$$V(C_5) = \{1, 2, 3, 4, 5\}, \qquad E(C_5) = \{12, 23, 34, 45, 51\}$$

3-GRAPH CYCLE LINEAR C_5^3:

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- Mubayi–Wang [2017+]: What is the number of C_5^3-free hypergraphs?

$$2^{\Theta(n^2)}?$$

Could the hypergraph container method be applied?

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- A hypergraph with $100n^2$ edges could contain many C_5's, but

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  True in similar problems: extremal bounds via balanced supersaturation carries to same bound on counting problem:

$$\mathrm{ex}(n, H) = O(n^\alpha) \quad \to \quad \text{number of } H\text{-free graphs} = 2^{O(n^\alpha)}.$$

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Definition

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- Lower bound: Give a lower bound on independence number of these (geometric) hypergraphs.

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- "Improves on all all applications of HCM, where it is applied to hypergraphs with large uniformicity".

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$\mathcal{N}$ point set in the plane, $\mathcal{L}$ line set in the plane. $X \subset \mathcal{N}$ is an ε-net if for every $\ell \in \mathcal{L}$ with $|\ell| \geqslant \varepsilon|\mathcal{N}|$ we have $X \cap \ell \neq \emptyset$.

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- Alon [2012]: False, sometimes $\omega\left(\frac{1}{\varepsilon}\log^*(\frac{1}{\varepsilon})\right)$ is a lower bound.

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Sometimes $\omega\left(\frac{1}{\varepsilon}\log^{1/3+o(1)}(\frac{1}{\varepsilon})\right)$ is a lower bound.

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- BSa new HCM works on both $[n]^2_p$ and $[n]^r_{1/n}$ giving for both ε-net of size
$$\frac{1}{\varepsilon}\log^{1/2+o(1)}\left(\frac{1}{\varepsilon}\right)$$

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- for every $C \in \mathcal{C}$: $\quad e(\mathcal{H}[C]) = o(e(\mathcal{H}))$.