

MINIMIZING THE NUMBER OF 5-CYCLES IN GRAPHS WITH GIVEN EDGE-DENSITY

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Workshop on graphs, networks, and their applications
Moscow, May 14 - 16, 2018

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$$\text{ex}(n, H) = \frac{\chi(H) - 2}{2(\chi(H) - 1)} n^2 + o(n^2).$$

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How many copies of H must be contained in a graph $G = (V, E)$ on n vertices and $|E(G)| > ex(n, H)$?

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$$d_H(G) = \frac{\nu_H(G)}{|V(G)||V(H)|}.$$

- For a given number $p \in [0, 1]$ let

$$d_H(p) = \lim_{n \rightarrow \infty} \min_G d_H(G),$$

where the minimum is taken over all graphs G of order n and size $p\binom{n}{2}$, assuming the limit exists.

SOME OF THE BREAKTHROUGH RESULTS

Goodman, 1959; Moon and Moser, 1962; Nordhaus and Stewart, 1963:

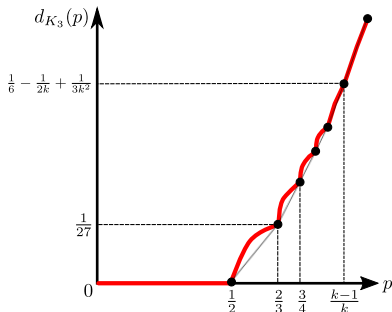
$$d_{K_3}(p) = \frac{1}{6} - \frac{1}{2k} + \frac{1}{3k^2} \text{ for } p = 1 - \frac{1}{k} \text{ and integer } k \geq 2$$

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Liu, Pikhurko, Staden, 2017+: Determined *exactly* the minimum number of triangles in a graph of given order and size.

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- $d_{K_r}(p)$ for any $r \geq 3$ and $p \in [0, 1]$ is well-understood.
- $d_{C_{2\ell}}(p)$ are also well-understood due to Sidorenko.

What about $d_{C_5}(p)$?

MAIN RESULT

THEOREM (BENNETT, D, LIDICKÝ, 2018)

Let $k \geq 2$ be an integer and $p = 1 - \frac{1}{k}$. Then,

$$d_{C_5}(p) = \frac{1}{10} - \frac{1}{2k} + \frac{1}{k^2} - \frac{1}{k^3} + \frac{2}{5k^4}.$$

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This bound is optimal, since a matching upper bound is given by the Turán graph T_n^k (the balanced complete k -partite graph of order n).

Proof is based on the flag algebras framework but actually can be verified without computer.


STABILITY

THEOREM (BDL, 2018)

Let G be a graph on n vertices for large n , such that G has edge density $p = 1 - \frac{1}{k}$ for $k \geq 2$ and

$$d_{C_5}(G) \leq d_{C_5}(p) + \varepsilon$$

for some positive but sufficiently small ε . Assume further that the only induced subgraphs on five vertices with density more than ε are the graphs


in: . Then G has edit distance at most δn^2 from the Turán graph T_n^k , for some function $\delta = \delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

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Proof is based on Induced Graph Removal Lemma and some technical optimizations.


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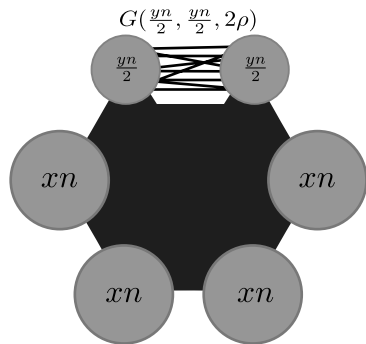
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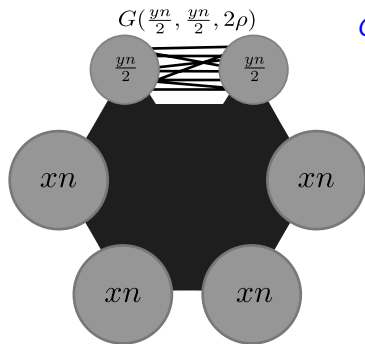
Let $k \in \{2, \dots, 73\}$. Then the assumption about non-zero induced subgraph densities hold. *And so the graphs with density $p = 1 - \frac{1}{k}$ that minimize the number of copies of C_5 are “close” to the Turán graph.*

GENERAL CASE: $1 - \frac{1}{k} < \rho < 1 - \frac{1}{k+1}$



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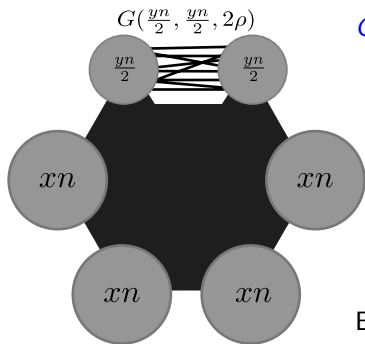


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C_5 -density: $f(x, y, \rho) = \lim_{n \rightarrow \infty} \frac{\nu_G(G_5)}{n^5}$

$$\begin{aligned} f(x, y, \rho) = & \left[\frac{1}{10}(k-1)_5 + \frac{1}{2}(k-1)_4 + \frac{1}{2}(k-1)_3 \right] x^5 \\ & + \left[\frac{1}{2}(k-1)_4 + \frac{3}{2}(k-1)_3 + \frac{1}{2}(k-1)_2 \right] x^4 y \\ & + \left[\left(\frac{1}{2} + \frac{1}{2}\rho \right) (k-1)_3 + \left(1 + \frac{1}{2}\rho \right) (k-1)_2 \right] x^3 y^2 \\ & + \left[\left(\frac{1}{2}\rho + \frac{1}{2}\rho^2 \right) (k-1)_2 + \frac{1}{2}\rho(k-1) \right] x^2 y^3 \\ & + \frac{1}{2}\rho^3(k-1)xy^4. \end{aligned}$$

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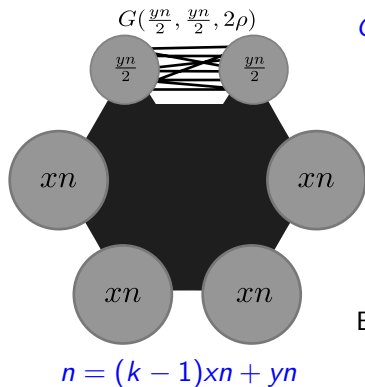
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Edge density: $g(x, y, \rho) = \lim_{n \rightarrow \infty} \frac{|E(G)|}{\binom{n}{2}}$

$$g(x, y, \rho) = (k-1)_2 x^2 + 2(k-1)xy + \rho y^2$$

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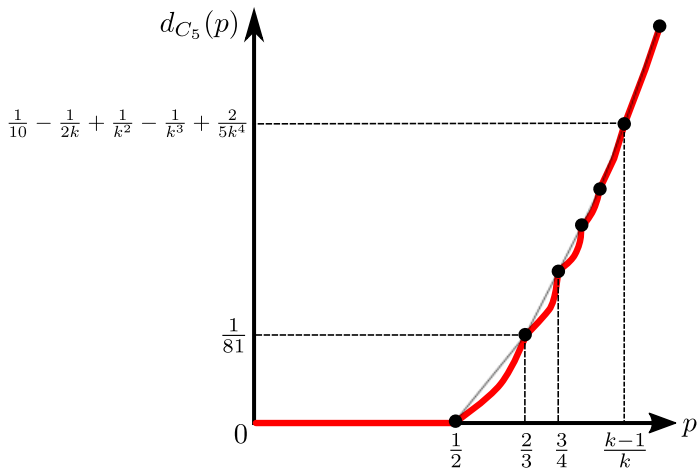
Minimize $f(x, y, \rho)$

subject to: $(k-1)x + y = 1,$

$$g(x, y, \rho) = \rho,$$

$$x, y \geq 0.$$

NUMERICAL RESULTS



FLAG ALGEBRAS

Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

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Features:

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

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A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2$ edges.

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We will use colors for **edges** and **non-edges**.

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Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\# \triangle / \binom{n}{3}$.

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The probability that a random vertex other than v is connected to $v \in V(G)$ by a red edge, i.e., the red degree of v divided by $n - 1$.

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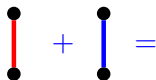
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$$\text{red edge} + \text{blue edge} = 1$$

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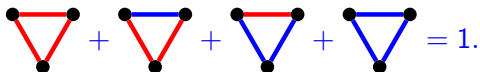
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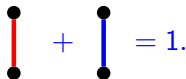
Flag

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then


$$\begin{array}{c} \bullet & & \bullet \\ \diagdown & & / \\ & \bullet & \\ \diagup & & \diagdown \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ \diagdown & & / \\ & \bullet & \\ \diagup & & \diagdown \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ \diagdown & & / \\ & \bullet & \\ \diagup & & \diagdown \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ \diagdown & & / \\ & \bullet & \\ \diagup & & \diagdown \\ \bullet & & \bullet \end{array} = 1.$$

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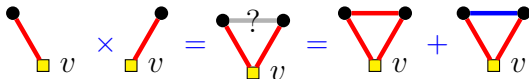
FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then by the law of total probability

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{3}{3} \begin{array}{c} \bullet \\ \color{red}{\diagup} \color{red}{\diagdown} \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \color{red}{\diagup} \color{blue}{\diagdown} \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \color{blue}{\diagup} \color{red}{\diagdown} \\ \bullet \end{array} + \frac{0}{3} \begin{array}{c} \bullet \\ \color{blue}{\diagup} \color{blue}{\diagdown} \\ \bullet \end{array} .$$

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The diagram shows an equation between graph structures. On the left, two flags are multiplied. Each flag consists of a yellow square vertex labeled v at the bottom, connected by red edges to two black circular vertices above it. The multiplication symbol \times is between them. This is followed by an equals sign and a flag with a grey edge between the two top vertices, with a question mark above it. This is followed by another equals sign and the sum of two flags: one with all red edges, and one with a blue edge between the top two vertices and red edges to the bottom vertex.



: The probability of choosing two **different** vertices ...

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: The probability of choosing two **different** vertices ...

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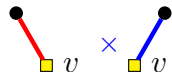
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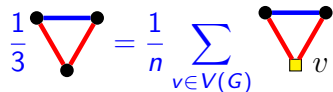
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: The probability that choosing two vertices u_1, u_2 other than v gives red vu_1 and blue vu_2 .

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Let G be a 2-edge-colored complete graph on n vertices. Then

$$\frac{1}{3} \text{ (triangle with 2 blue edges) } = \frac{1}{n} \sum_{v \in V(G)} \text{ (triangle with 2 blue edges and vertex } v \text{ highlighted)}$$
The diagram shows an equation between two graph structures. On the left, a triangle with three vertices and three edges. The top edge is blue, and the two bottom edges are red. This is multiplied by the fraction 1/3. This is equal to the fraction 1/n multiplied by a summation over all vertices v in the vertex set V(G). The summation term is a triangle with three vertices and three edges. The top edge is blue, and the two bottom edges are red. The bottom-right vertex is highlighted with a yellow square and labeled 'v'.

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then

$$\frac{1}{3} \text{ (triangle with 2 blue edges) } = \frac{1}{n} \sum_{v \in V(G)} \text{ (triangle with 2 blue edges and vertex } v \text{ highlighted)}$$

$$\text{ (triangle with 2 blue edges) } \binom{n}{3} = \sum_{v \in V(G)} \text{ (triangle with 2 blue edges and vertex } v \text{ highlighted) } \binom{n-1}{2}$$

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Let G be a 2-edge-colored complete graph on n vertices. Then

$$\frac{1}{3} \text{ (triangle with 2 blue edges) } = \frac{1}{n} \sum_{v \in V(G)} \text{ (triangle with 2 blue edges, vertex } v \text{ highlighted)}$$

$$\text{ (triangle with 3 red edges) } = \frac{1}{n} \sum_{v \in V(G)} \text{ (triangle with 3 red edges, vertex } v \text{ highlighted)}$$

$$\text{ (triangle with 2 blue edges) } \binom{n}{3} = \sum_{v \in V(G)} \text{ (triangle with 2 blue edges, vertex } v \text{ highlighted) } \binom{n-1}{2}$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then

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$$\text{ (triangle with 3 red edges) } \binom{n}{3} = \frac{1}{3} \sum_{v \in V(G)} \text{ (triangle with 3 red edges, vertex } v \text{ highlighted) } \binom{n-1}{2}$$

IDENTITIES SUMMARY

Let G be a 2-edge-colored complete graph on n vertices. Then

$$1 = \begin{array}{c} \bullet \\ \diagdown \text{ (red) } \\ \bullet \\ \diagup \text{ (red) } \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \text{ (blue) } \\ \bullet \\ \diagup \text{ (red) } \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \text{ (red) } \\ \bullet \\ \diagup \text{ (blue) } \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \text{ (blue) } \\ \bullet \\ \diagup \text{ (blue) } \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{3}{3} \begin{array}{c} \bullet \\ \diagdown \text{ (red) } \\ \bullet \\ \diagup \text{ (red) } \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \diagdown \text{ (blue) } \\ \bullet \\ \diagup \text{ (red) } \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \text{ (red) } \\ \bullet \\ \diagup \text{ (blue) } \\ \bullet \end{array} + \frac{0}{3} \begin{array}{c} \bullet \\ \diagdown \text{ (blue) } \\ \bullet \\ \diagup \text{ (blue) } \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} \times \begin{array}{c} \bullet \\ | \\ \square v \end{array} = \begin{array}{c} \bullet \\ \diagdown \text{ (red) } \\ \square v \\ \diagup \text{ (red) } \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \text{ (blue) } \\ \square v \\ \diagup \text{ (red) } \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} \times \begin{array}{c} \bullet \\ | \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \\ \diagdown \text{ (red) } \\ \square v \\ \diagup \text{ (blue) } \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \diagdown \text{ (blue) } \\ \square v \\ \diagup \text{ (blue) } \\ \bullet \end{array}$$

$$\frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \text{ (blue) } \\ \bullet \\ \diagup \text{ (red) } \\ \bullet \end{array} = \frac{1}{n} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \diagdown \text{ (red) } \\ \square v \\ \diagup \text{ (red) } \\ \bullet \end{array}$$

and

$$\begin{array}{c} \bullet \\ \diagdown \text{ (red) } \\ \bullet \\ \diagup \text{ (red) } \\ \bullet \end{array} = \frac{1}{n} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \diagdown \text{ (red) } \\ \square v \\ \diagup \text{ (red) } \\ \bullet \end{array}$$

MANTEL'S THEOREM - 1ST TRY

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.



Assume **edges are red** and **non-edges are blue**.

MANTEL'S THEOREM - 1ST TRY

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A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

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Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)





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$$\text{edge} = 0 \cdot \text{triangle} + \frac{1}{3} \cdot \text{triangle} + \frac{2}{3} \cdot \text{triangle}$$
 = 0  + $\frac{1}{3}$  + $\frac{2}{3}$ 

MANTEL'S THEOREM - 1ST TRY

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A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

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$$\begin{aligned} \text{Edge} &= 0 \cdot \text{Triangle} + \frac{1}{3} \cdot \text{Triangle} + \frac{2}{3} \cdot \text{Triangle} \\ &\leq \frac{2}{3} \left(\text{Triangle} + \text{Triangle} + \text{Triangle} \right) \end{aligned}$$

MANTEL'S THEOREM - 1ST TRY

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$$\begin{aligned} \text{Edge} &= 0 \cdot \text{Triangle} + \frac{1}{3} \cdot \text{Triangle} + \frac{2}{3} \cdot \text{Triangle} \\ &\leq \frac{2}{3} \left(\text{Triangle} + \text{Triangle} + \text{Triangle} \right) \\ 1 &= \text{Triangle} + \text{Triangle} + \text{Triangle} + \text{Triangle} \end{aligned}$$

MANTEL'S THEOREM - 1ST TRY

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

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$$\begin{aligned} \text{edge} &= 0 \cdot \text{triangle} + \frac{1}{3} \cdot \text{triangle} + \frac{2}{3} \cdot \text{triangle} \\ &\leq \frac{2}{3} \left(\text{triangle} + \text{triangle} + \text{triangle} \right) \\ &= 1 \cdot \text{triangle} + \text{triangle} + \text{triangle} \end{aligned}$$

MANTEL'S THEOREM - 1ST TRY

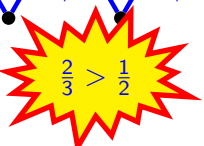
THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

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Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\begin{aligned} \text{edge} &= 0 \cdot \text{triangle} + \frac{1}{3} \cdot \text{triangle} + \frac{2}{3} \cdot \text{triangle} \\ &\leq \frac{2}{3} \left(\text{triangle} + \text{triangle} + \text{triangle} \right) \\ &= 1 \cdot \text{triangle} + \text{triangle} + \text{triangle} \\ &\leq \frac{2}{3} \end{aligned}$$



MANTEL'S THEOREM - 2ND TRY

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.



Assume edges are red and non-edges are blue.

MANTEL'S THEOREM - 2ND TRY

THEOREM (MANTEL 1907)

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$$0 \leq \left(1 - 2 \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^2$$





MANTEL'S THEOREM - 2ND TRY

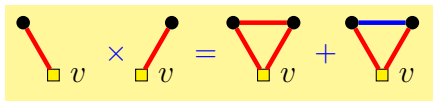
THEOREM (MANTEL 1907)





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Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \left(1 - 2 \text{  } v \right)^2 = \left(1 - 4 \text{  } v + 4 \text{  } v + 4 \text{  } v \right)$$





$$\text{  } v \times \text{  } v = \text{  } v + \text{  } v$$

MANTEL'S THEOREM - 2ND TRY

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

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$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{img alt="red edge with vertex v" data-bbox="281 453 301 558"} \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{img alt="red edge with vertex v" data-bbox="588 458 608 558"} + 4 \cdot \text{img alt="red triangle with vertex v" data-bbox="698 458 773 558"} + 4 \cdot \text{img alt="red triangle with vertex v" data-bbox="833 458 908 558"} \right)$$

MANTEL'S THEOREM - 2ND TRY

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{img alt="edge with 1 red edge and 1 blue edge" data-bbox="285 455 315 565"}_v \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{img alt="edge with 1 red edge and 1 blue edge" data-bbox="585 455 615 565"}_v + 4 \cdot \text{img alt="triangle with 2 red edges and 1 blue edge" data-bbox="695 455 775 565"}_v + 4 \cdot \text{img alt="triangle with 3 red edges" data-bbox="835 455 915 565"}_v \right)$$

$$= 1 - 4 \cdot \text{img alt="edge with 1 red edge and 1 blue edge" data-bbox="195 595 215 695"} + \frac{4}{3} \cdot \text{img alt="triangle with 2 red edges and 1 blue edge" data-bbox="295 595 375 695"} + 4 \cdot \text{img alt="triangle with 3 red edges" data-bbox="425 595 505 695"}$$

$$\frac{1}{3} \cdot \text{img alt="triangle with 2 red edges and 1 blue edge" data-bbox="95 835 175 935"} = \frac{1}{n} \sum_{v \in V(G)} \text{img alt="triangle with 2 red edges and 1 blue edge" data-bbox="355 835 435 935"}_v$$



$$\text{img alt="triangle with 3 red edges" data-bbox="575 835 655 935"} = \frac{1}{n} \sum_{v \in V(G)} \text{img alt="triangle with 3 red edges" data-bbox="835 835 915 935"}_v$$





MANTEL'S THEOREM - 2ND TRY




THEOREM (MANTEL 1907)





A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \text{}_v \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \text{}_v + 4 \text{}_v + 4 \text{}_v \right)$$

$$= 1 - 4 \text{} + \frac{4}{3} \text{} + 4 \text{}$$



$$\text{} = \frac{2}{3} \text{} + \frac{1}{3} \text{} + \text{}$$

MANTEL'S THEOREM - 2ND TRY

THEOREM (MANTEL 1907)




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Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{img alt="red edge with yellow square at v" data-bbox="278 455 315 565"} \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{img alt="red edge with yellow square at v" data-bbox="588 455 625 565"} + 4 \cdot \text{img alt="red triangle with yellow square at v" data-bbox="695 455 772 565"} + 4 \cdot \text{img alt="red triangle with yellow square at v" data-bbox="832 455 909 565"} \right)$$

$$= 1 - 4 \cdot \text{img alt="red edge" data-bbox="198 595 215 692"} + \frac{4}{3} \cdot \text{img alt="red triangle" data-bbox="298 595 375 692"}$$



 = $\frac{2}{3}$  + $\frac{1}{3}$ 

MANTEL'S THEOREM - 2ND TRY

THEOREM (MANTEL 1907)

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Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{img alt="red edge with yellow square at v" data-bbox="275 455 315 565"} \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{img alt="red edge with yellow square at v" data-bbox="585 455 625 565"} + 4 \cdot \text{img alt="red triangle with yellow square at v" data-bbox="695 455 775 565"} + 4 \cdot \text{img alt="red triangle with yellow square at v" data-bbox="835 455 915 565"} \right)$$

$$= 1 - 4 \cdot \text{img alt="red edge" data-bbox="195 595 215 695"} + \frac{4}{3} \cdot \text{img alt="red triangle" data-bbox="295 595 375 695"}$$

$$0 = 2 \cdot \text{img alt="red edge" data-bbox="555 645 575 745"} - \frac{4}{3} \cdot \text{img alt="red triangle" data-bbox="655 645 735 745"} - \frac{2}{3} \cdot \text{img alt="red triangle" data-bbox="785 645 865 745"}$$



$$\text{img alt="red edge" data-bbox="435 805 455 905"} = \frac{2}{3} \cdot \text{img alt="red triangle" data-bbox="535 805 615 905"} + \frac{1}{3} \cdot \text{img alt="red triangle" data-bbox="665 805 745 905"}$$

MANTEL'S THEOREM - 2ND TRY

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$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{img alt="edge with 1 red edge and 1 blue edge" data-bbox="585 455 615 565"}_v \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{img alt="edge with 1 red edge and 1 blue edge" data-bbox="585 455 615 565"}_v + 4 \cdot \text{img alt="triangle with 2 red edges and 1 blue edge" data-bbox="695 455 775 565"}_v + 4 \cdot \text{img alt="triangle with 1 red edge and 2 blue edges" data-bbox="825 455 905 565"}_v \right)$$

$$= 1 - 4 \cdot \text{img alt="edge with 1 red edge and 1 blue edge" data-bbox="195 595 215 715"} + \frac{4}{3} \cdot \text{img alt="triangle with 2 red edges and 1 blue edge" data-bbox="295 595 375 685"} - \frac{2}{3} \cdot \text{img alt="triangle with 1 red edge and 2 blue edges" data-bbox="295 715 375 805"}$$

$$= 1 - 2 \cdot \text{img alt="edge with 1 red edge and 1 blue edge" data-bbox="195 715 215 815"} - \frac{2}{3} \cdot \text{img alt="triangle with 1 red edge and 2 blue edges" data-bbox="295 715 375 805"}$$

$$0 = 2 \cdot \text{img alt="edge with 1 red edge and 1 blue edge" data-bbox="555 645 575 745"} - \frac{4}{3} \cdot \text{img alt="triangle with 2 red edges and 1 blue edge" data-bbox="655 645 735 735"} - \frac{2}{3} \cdot \text{img alt="triangle with 1 red edge and 2 blue edges" data-bbox="785 645 865 735"}$$



$$\text{img alt="edge with 1 red edge and 1 blue edge" data-bbox="435 805 455 905"} = \frac{2}{3} \cdot \text{img alt="triangle with 2 red edges and 1 blue edge" data-bbox="535 805 615 895"} + \frac{1}{3} \cdot \text{img alt="triangle with 1 red edge and 2 blue edges" data-bbox="665 805 745 895"}$$

MANTEL'S THEOREM - 2ND TRY

THEOREM (MANTEL 1907)

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Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="588 455 605 555"}_v \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="588 455 605 555"}_v + 4 \cdot \text{img alt="red triangle with yellow square at bottom vertex" data-bbox="695 455 772 555"}_v + 4 \cdot \text{img alt="red triangle with yellow square at bottom vertex" data-bbox="828 455 905 555"}_v \right)$$

$$= 1 - 4 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="198 595 215 685"} + \frac{4}{3} \cdot \text{img alt="red triangle with yellow square at bottom vertex" data-bbox="295 595 372 685"/>$$

$$= 1 - 2 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="198 718 215 808"} - \frac{2}{3} \cdot \text{img alt="red triangle with yellow square at bottom vertex" data-bbox="295 718 372 808"/>$$

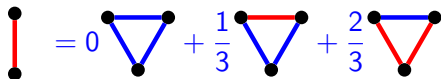
$$\leq 1 - 2 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="198 864 215 934"} . \quad \square$$

$$0 = 2 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="558 646 575 736"} - \frac{4}{3} \cdot \text{img alt="red triangle with yellow square at bottom vertex" data-bbox="655 646 732 736"} - \frac{2}{3} \cdot \text{img alt="red triangle with yellow square at bottom vertex" data-bbox="788 646 865 736"/>$$

$$\text{img alt="red edge with yellow square at bottom vertex" data-bbox="438 806 455 896"} = \frac{2}{3} \cdot \text{img alt="red triangle with yellow square at bottom vertex" data-bbox="535 806 612 896"} + \frac{1}{3} \cdot \text{img alt="red triangle with yellow square at bottom vertex" data-bbox="668 806 745 896"/>$$

HOW TO GUESS $0 \leq \left(1 - 2 \underset{\substack{\bullet \\ | \\ \square}}{v}}\right)^2 ?$

HOW TO GUESS $0 \leq \left(1 - 2 \underset{\text{yellow square } v}{\overset{\text{black dot}}{\mid}}\right)^2$?

$$\text{red edge} = 0 \cdot \text{blue triangle} + \frac{1}{3} \cdot \text{red-blue triangle} + \frac{2}{3} \cdot \text{red triangle}$$


HOW TO GUESS $0 \leq \left(1 - 2 \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^2$?

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} = 0 \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$0 \leq \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T$$

$$= c_1 \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + c_2 \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + c_3 \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + c_4 \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array},$$

$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is a positive semidefinite matrix

HOW TO GUESS $0 \leq \left(1 - 2 \begin{array}{c} \bullet \\ | \\ \square v \end{array}\right)^2$?

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} = 0 \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array}$$

$$0 \leq \begin{pmatrix} \begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \end{pmatrix}^T$$

$$= c_1 \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + c_2 \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + c_3 \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + c_4 \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array},$$

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} \leq \max \left\{ 0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\} \underbrace{\left(\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ \diagup \\ \bullet \end{array} \right)}_{=1}$$

$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is a positive semidefinite matrix, can be optimized.

CANDIDATES FOR c_1, c_2, c_3

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$0 \leq \left(\begin{array}{c|c} \bullet & \bullet \\ \hline \color{blue}{\downarrow} & \color{red}{\downarrow} \\ \color{yellow}{\square} & \color{yellow}{\square} \end{array} \begin{array}{c} v \\ v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c|c} \bullet & \bullet \\ \hline \color{blue}{\downarrow} & \color{red}{\downarrow} \\ \color{yellow}{\square} & \color{yellow}{\square} \end{array} \begin{array}{c} v \\ v \end{array} \right)^T$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \quad ? \\ \diagdown \quad / \\ \square v \end{array} + b \begin{array}{c} \bullet \quad ? \\ \diagup \quad \diagdown \\ \square v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \\ \diagdown \quad / \\ \square v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \\ \diagup \quad \diagdown \\ \square v \end{array}
 \end{aligned}$$

$$\begin{array}{c} \bullet \\ \diagdown \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \diagup \\ \square v \end{array} = \begin{array}{c} \bullet \quad ? \\ \diagdown \quad / \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \\ \diagdown \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \diagup \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \quad ? \\ \diagdown \quad / \\ \square v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ / \quad \backslash \\ \square v \end{array} + b \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ \backslash \quad / \\ \square v \end{array} + c \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ \backslash \quad / \\ \square v \end{array}
 \end{aligned}$$

$$\begin{array}{c} \bullet \\ \backslash \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \backslash \\ \square v \end{array} = \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ \backslash \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \\ \backslash \\ \square v \end{array} \times \begin{array}{c} \bullet \\ / \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ \backslash \\ \square v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$0 \leq \begin{pmatrix} \begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \end{pmatrix}^T$$
$$= \begin{array}{c} \bullet \\ \text{?} \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square v \end{array} \quad + \quad \begin{array}{c} \bullet \\ \text{?} \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square v \end{array} \quad + \quad \begin{array}{c} \bullet \\ \text{?} \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square v \end{array}$$

The diagram shows the expansion of a quadratic form. The first term is a vector of two vertical lines (one blue, one red) with a yellow square at the bottom labeled 'v' and a black dot at the top. This is multiplied by a 2x2 matrix with entries 'a', 'c', 'c', and 'b'. This is then multiplied by the transpose of the first vector. The result is shown as the sum of three terms, each with a gray horizontal line and a question mark above it, and a yellow square at the bottom labeled 'v'. The first term has two blue lines connecting the top dots to the square. The second term has two red lines. The third term has one red line and one blue line.

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + b \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \square v \end{array} + c \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \begin{pmatrix} \bullet & \bullet \\ \square_v & \square_v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \bullet & \bullet \\ \square_v & \square_v \end{pmatrix}^T \\
 &= \frac{1}{n} \sum_v a \begin{matrix} \bullet & ? \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} + b \begin{matrix} \bullet & ? \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} + c \begin{matrix} \bullet & ? \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} \\
 &= a \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} + \frac{a+2c}{3} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} + \frac{b+2c}{3} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} + b \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix}
 \end{aligned}$$

$$\frac{1}{3} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix}$$

$$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \left(\frac{2}{3} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} \right)$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \begin{pmatrix} \bullet & \bullet \\ \square_v & \square_v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \bullet & \bullet \\ \square_v & \square_v \end{pmatrix}^T \\
 &= \frac{1}{n} \sum_v a \begin{matrix} \bullet & ? \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} + b \begin{matrix} \bullet & ? \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} + c \begin{matrix} \bullet & ? \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} \\
 &= a \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} + \frac{a+2c}{3} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} + \frac{b+2c}{3} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix}
 \end{aligned}$$

$$\frac{1}{3} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix}$$

$$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \left(\frac{2}{3} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \square_v & \bullet \end{matrix} \right)$$

CANDIDATES FOR c_1, c_2, c_3

$$0 \leq \frac{1}{n} \sum_v \begin{pmatrix} \bullet \\ \square v \end{pmatrix}, \begin{pmatrix} \bullet \\ \square v \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \bullet \\ \square v \end{pmatrix}, \begin{pmatrix} \bullet \\ \square v \end{pmatrix} \begin{pmatrix} \bullet \\ \square v \end{pmatrix}^T$$

$$= \frac{1}{n} \sum_v a \begin{matrix} \bullet & ? & \bullet \\ & \diagdown & / \\ & \square v & \end{matrix} + b \begin{matrix} \bullet & ? & \bullet \\ & / & \diagdown \\ & \square v & \end{matrix} + c \begin{matrix} \bullet & ? & \bullet \\ & \diagdown & / \\ & \square v & \end{matrix}$$

$$= a \begin{matrix} \bullet & & \bullet \\ & \diagdown & / \\ & \square v & \end{matrix} + \frac{a+2c}{3} \begin{matrix} \bullet & & \bullet \\ & \diagdown & / \\ & \square v & \end{matrix} + \frac{b+2c}{3} \begin{matrix} \bullet & & \bullet \\ & / & \diagdown \\ & \square v & \end{matrix}$$

$$c_1 = a, \quad c_2 = \frac{a+2c}{3}, \quad c_3 = \frac{b+2c}{3}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}$$

USING c_1, c_2, c_3

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \\ 0 \leq a &+ \frac{a+2c}{3} + \frac{b+2c}{3} \end{aligned}$$

The diagram shows a vertical red line segment on the left. To its right is an equals sign followed by three terms. The first term is a blue triangle with vertices at the top-left, top-right, and bottom. The second term is a fraction 1/3 multiplied by a triangle with a red top edge and blue sides. The third term is a fraction 2/3 multiplied by a triangle with a red right edge and blue left and bottom edges. Below this is a less-than-or-equal-to sign followed by three terms: a blue triangle, a fraction (a+2c)/3 multiplied by a triangle with a red top edge and blue sides, and a fraction (b+2c)/3 multiplied by a triangle with a red right edge and blue left and bottom edges.

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}$$

USING c_1, c_2, c_3

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \\ 0 \leq a &+ \frac{a+2c}{3} + \frac{b+2c}{3} \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}$$

USING c_1, c_2, c_3

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} \\ 0 \leq a &+ \frac{a+2c}{3} + \frac{b+2c}{3} \end{aligned}$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}.$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \text{ (matrix is positive semidefinite)}$$

USING c_1, c_2, c_3

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ | \\ \bullet \end{array} \\ 0 \leq a &+ \frac{a+2c}{3} + \frac{b+2c}{3} \end{aligned}$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}.$$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

USING c_1, c_2, c_3

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} \\ 0 \leq a &+ \frac{a+2c}{3} + \frac{b+2c}{3} \end{aligned}$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}.$$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}.$$

OPTIMIZING a, b, c

$$\| \cdot \| \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}$$

$$(SDP) \left\{ \begin{array}{ll} \text{Minimize} & d \\ \text{subject to:} & a \leq d \\ & \frac{1+a+2c}{3} \leq d \\ & \frac{2+b+2c}{3} \leq d \\ & \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \end{array} \right.$$

(SDP) can be solved on computers using CSDP or SDPA.
Rounding may be needed for exact results using SAGE.

PROVING THE MAIN RESULT

THEOREM (BENNETT, D, LIDICKÝ, 2018)

Let $k \geq 2$ be an integer and $p = 1 - \frac{1}{k}$. Then,

$$d_{C_5}(p) = \frac{1}{10} - \frac{1}{2k} + \frac{1}{k^2} - \frac{1}{k^3} + \frac{2}{5k^4}.$$

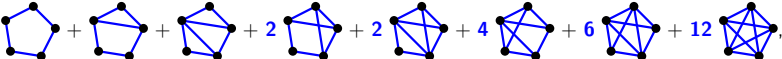
PROVING THE MAIN RESULT

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Let

$$OPT = \text{graph}_1 + \text{graph}_2 + \text{graph}_3 + 2 \text{graph}_4 + 2 \text{graph}_5 + 4 \text{graph}_6 + 6 \text{graph}_7 + 12 \text{graph}_8,$$


where the coefficient of each graph is the # of copies of C_5 it contains.

PROVING THE MAIN RESULT

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Let $k \geq 2$ be an integer and $p = 1 - \frac{1}{k}$. Then,

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Let

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where the coefficient of each graph is the # of copies of C_5 it contains.

Minimize

subject to:

$$\text{graph}_1 \geq p$$

BOUNDING OPT

Let

$$X = \left(\begin{array}{c} \bullet \\ \square \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \square \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \square \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \square \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \square \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \square \end{array} \right)^T.$$

Then, $\llbracket X^T M X \rrbracket \geq 0$ for any positive semidefinite matrix $M = M_{6 \times 6}$ and it is a linear combination of all graphs on 5 vertices (denoted by \mathcal{F}_5).

BOUNDING *OPT*

Let

$$X = \left(\begin{array}{c} \bullet \\ \square \\ \bullet, \square \\ \bullet, \square, \bullet \\ \bullet, \square, \bullet, \square \\ \bullet, \square, \bullet, \square, \bullet \\ \bullet, \square, \bullet, \square, \bullet, \square \end{array} \right)^T.$$

Then, $\llbracket X^T M X \rrbracket \geq 0$ for any positive semidefinite matrix $M = M_{6 \times 6}$ and it is a linear combination of all graphs on 5 vertices (denoted by \mathcal{F}_5).

Also,

$$\begin{array}{c} \bullet \\ \bullet \end{array} = \frac{1}{10} \begin{array}{c} \bullet \\ \bullet, \bullet \\ \bullet, \bullet \\ \bullet, \bullet \end{array} + \frac{1}{5} \begin{array}{c} \bullet \\ \bullet, \bullet, \bullet \\ \bullet, \bullet, \bullet \\ \bullet, \bullet, \bullet \end{array} + \frac{3}{10} \begin{array}{c} \bullet \\ \bullet, \bullet, \bullet, \bullet \\ \bullet, \bullet, \bullet, \bullet \\ \bullet, \bullet, \bullet, \bullet \end{array} + \dots + \begin{array}{c} \bullet \\ \bullet, \bullet, \bullet, \bullet, \bullet \\ \bullet, \bullet, \bullet, \bullet, \bullet \\ \bullet, \bullet, \bullet, \bullet, \bullet \end{array}$$

BOUNDING OPT

Let

$$X = \left(\begin{array}{c} \bullet \\ \square \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right)^T.$$

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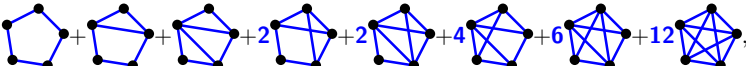





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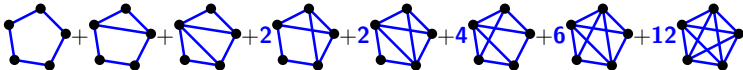
and

$$p \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = p \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + p \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + p \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} + \dots + p \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

BOUNDING OPT

Since $OPT =$  + 2  + 2  + 4  + 6  + 12 ,

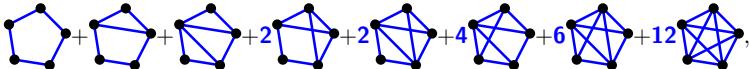
BOUNDING OPT

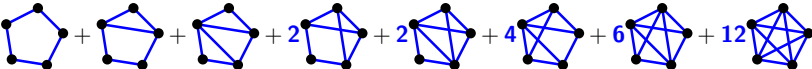

Since $OPT =$  ,

we get

$$OPT \geq \text{pentagon} + \text{pentagon} + \text{pentagon} + 2 \text{pentagon} + 2 \text{pentagon} + 4 \text{pentagon} + 6 \text{pentagon} + 12 \text{pentagon} \\ - \alpha \left(\text{edge} - p \right) - \llbracket X^T M X \rrbracket$$

BOUNDING OPT

Since $OPT =$ ,
we get

$$\begin{aligned} OPT &\geq \text{} \\ &\quad - \alpha \left(\text{} - p \right) - \llbracket X^T M X \rrbracket \\ &= \sum_{F \in \mathcal{F}_5} c_F \cdot F \geq \min_{F \in \mathcal{F}_5} c_F \cdot \sum_{F \in \mathcal{F}_5} F = \min_{F \in \mathcal{F}_5} c_F. \end{aligned}$$

Clearly, each $c_F = c_F(M, \alpha)$.

OPTIMIZING COEFFICIENTS C_F

In order to define matrix M we define first two matrices A and B :

$$A = \begin{pmatrix} 32k^2 - 96k + 96 & 0 & 4k^2 - 16k \\ 0 & 10k^4 - 30k^3 - 8k^2 + 96k - 96 & -10k^4 + 35k^3 - 4k^2 - 80k + 96 \\ 4k^2 - 16k & -10k^4 + 35k^3 - 4k^2 - 80k + 96 & 10k^4 - 40k^3 + 24k^2 + 64k - 96 \end{pmatrix}$$

and

$$B = \begin{pmatrix} k-1 & 1 & k-2 & 0 & k-3 & -1 \\ 0 & 2 & k-2 & 0 & 2k-4 & -2 \\ 0 & 0 & k-1 & -1 & 2k-2 & -2 \end{pmatrix}.$$

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It is easy to verify (by checking principal minors) that A is positive definite for any $k \geq 3$. Therefore, matrix

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Set

$$\alpha = \frac{1}{600k^4} (30k^3 - 120k^2 + 180k - 96).$$

OPTIMIZING COEFFICIENTS c_F

$$\begin{aligned}
 c_{\cdot\cdot} &= c_{\nearrow} = c_{\nwarrow} = c_{\nearrow} = c_{\nwarrow} = c_{\nearrow} = c_{\nwarrow} = c_{\nearrow} = c_{\nwarrow} = c_{\nearrow} = c_{\nwarrow} = c_{\nearrow} = c_{\nwarrow} = \\
 c_{\nearrow} &= c_{\nwarrow} = c_{\nearrow} = c_{\nwarrow} = c_{\nearrow} = c_{\nwarrow} = \frac{1}{600k^4}(60k^4 - 300k^3 + 600k^2 - 600k + 240) \\
 c_{\cdot\cdot} &= c_{\nearrow} = c_{\nwarrow} = c_{\nearrow} = c_{\nwarrow} = \frac{1}{600k^4}(66k^4 - 300k^3 + 600k^2 - 600k + 240) \\
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 \end{aligned}$$

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$$\begin{aligned}
 c_{\cdot\cdot} &= c_{\nearrow} = c_{\nwarrow} = c_{\searrow} = c_{\swarrow} = c_{\nearrow} = c_{\nwarrow} = c_{\searrow} = c_{\swarrow} = c_{\nearrow} = c_{\nwarrow} = c_{\searrow} = c_{\swarrow} = \\
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 \end{aligned}$$

Thus,

$$OPT \geq \min_{F \in \mathcal{F}_5} c_F = \frac{1}{600k^4}(60k^4 - 300k^3 + 600k^2 - 600k + 240) = \frac{1}{10} - \frac{1}{2k} + \frac{1}{k^2} - \frac{1}{k^3} + \frac{2}{5k^4}.$$

STABILITY

Once again,

$$\begin{aligned} 12 - \frac{60}{k} + \frac{120}{k^2} - \frac{120}{k^3} + \frac{48}{k^4} &\geq \text{OPT} \geq \sum_{F \in \mathcal{F}_5} c_F \cdot F \geq \min_{F \in \mathcal{F}_5} c_F \cdot \sum_{F \in \mathcal{F}_5} F \\ &= \min_{F \in \mathcal{F}_5} c_F = 12 - \frac{60}{k} + \frac{120}{k^2} - \frac{120}{k^3} + \frac{48}{k^4}. \end{aligned}$$

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Thus, if $c_F > \min_{H \in \mathcal{F}_5} c_H$, then the density of F must be zero.

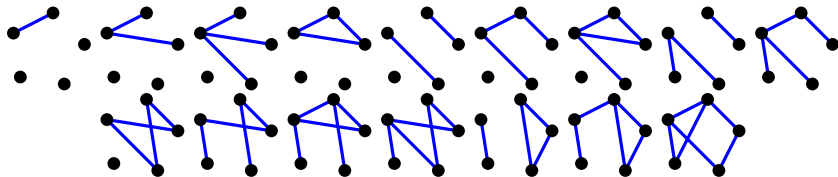
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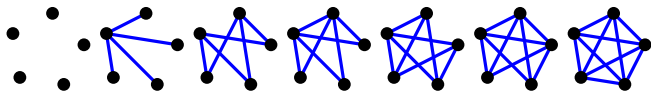
Thus, if $c_F > \min_{H \in \mathcal{F}_5} c_H$, then the density of F must be zero.

And so the following graphs have **zero** density in the limit:



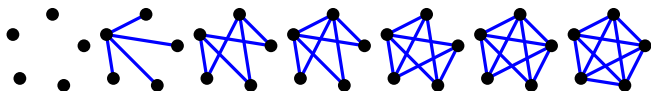
STABILITY

As a matter of fact for $2 \leq k \leq 73$ one can show by using flags with more labeled vertices that the only possible graphs with **nonzero** density must belong to the following list \mathcal{L} :



STABILITY

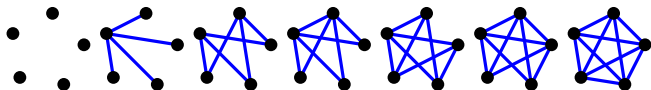
As a matter of fact for $2 \leq k \leq 73$ one can show by using flags with more labeled vertices that the only possible graphs with **nonzero** density must belong to the following list \mathcal{L} :



Apply to G the *induced graph removal lemma* to eliminate all induced subgraphs that are not in \mathcal{L} . Call this new graph G' . Observe that G' has no induced copy of \bullet and therefore is a complete partite graph.

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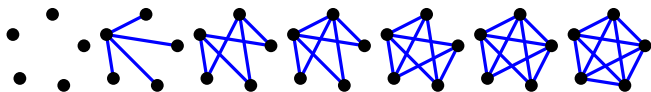


Apply to G the *induced graph removal lemma* to eliminate all induced subgraphs that are not in \mathcal{L} . Call this new graph G' . Observe that G' has no induced copy of \bullet and therefore is a complete partite graph.

Simple but tedious calculations show that G' is k -partite.

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Apply to G the induced graph removal lemma to eliminate all induced subgraphs that are not in \mathcal{L} . Call this new graph G' . Observe that G' has no induced copy of K_5 and therefore is a complete partite graph.

Simple but tedious calculations show that G' is k -partite.

And so the graphs with density $p = 1 - \frac{1}{k}$ that minimize the number of copies of C_5 are “close” to the Turán graph.

FURTHER DIRECTIONS

Determine $d_{C_5}(p)$ for $p \neq 1 - \frac{1}{k}$.

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General stability results.

Thanks!