Minimizing the number of 5-cycles in graphs with given edge-density

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How many copies of H must be contained in a graph $G = (V, E)$ on n vertices and $|E(G)| > ex(n, H)$?

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- Density of H in G
$$d_H(G) = \frac{\nu_H(G)}{|V(G)|^{|V(H)|}}.$$
- For a given number $p \in [0,1]$ let
$$d_H(p) = \lim_{n\to\infty} \min_G d_H(G),$$
where the minimum is taken over all graphs G of order n and size $p\binom{n}{2}$, assuming the limit exists.

Some of the breakthrough results

Goodman, 1959; Moon and Moser, 1962; Nordhaus and Stewart, 1963:

$$d_{K_3}(p) = \frac{1}{6} - \frac{1}{2k} + \frac{1}{3k^2} \text{ for } p = 1 - \frac{1}{k} \text{ and integer } k \geq 2$$

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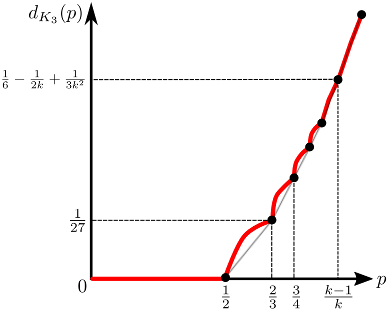

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Liu, Pikhurko, Staden, 2017+: Determined *exactly* the minimum number of triangles in a graph of given order and size.

More motivation

Summarizing:

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- $d_{C_{2\ell}}(p)$ are also well-understood due to Sidorenko.

What about $d_{C_5}(p)$?

Main result

Theorem (Bennett, D, Lidický, 2018)

Let $k \geq 2$ *be an integer and* $p = 1 - \frac{1}{k}$. *Then,*

$$d_{C_5}(p) = \frac{1}{10} - \frac{1}{2k} + \frac{1}{k^2} - \frac{1}{k^3} + \frac{2}{5k^4}.$$

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Proof is based on the flag algebras framework but actually can be verified without computer.

Stability

Theorem (BDL, 2018)

Let G be a graph on n vertices for large n, such that G has edge density $p = 1 - \frac{1}{k}$ for $k \geq 2$ and

$$d_{C_5}(G) \leq d_{C_5}(p) + \varepsilon$$

for some positive but sufficiently small ε. Assume further that the only induced *subgraphs on five vertices with density more than ε are the graphs in:* *. Then G has edit distance at most δn^2 from the Turán graph T_n^k, for some function $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

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Proof is based on Induced Graph Removal Lemma and some technical optimizations.

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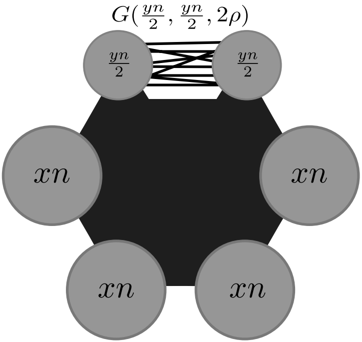
Theorem (BDL, 2018)

Let $k \in \{2, \ldots, 73\}$. Then the assumption about non-zero induced subgraph densities hold. And so the graphs with density $p = 1 - \frac{1}{k}$ that minimize the number of copies of C_5 are "close" to the Turán graph.

General case: $1 - \frac{1}{k} < p < 1 - \frac{1}{k+1}$



$n = (k-1)xn + yn$

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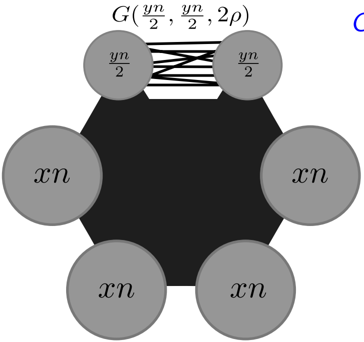

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C_5-density: $f(x, y, \rho) = \lim_{n\to\infty} \frac{\nu_G(C_5)}{n^5}$

$$
\begin{aligned}
f(x,y,\rho) = &\left[\frac{1}{10}(k-1)_5 + \frac{1}{2}(k-1)_4 + \frac{1}{2}(k-1)_3\right] x^5 \\
&+ \left[\frac{1}{2}(k-1)_4 + \frac{3}{2}(k-1)_3 + \frac{1}{2}(k-1)_2\right] x^4 y \\
&+ \left[\left(\frac{1}{2} + \frac{1}{2}\rho\right)(k-1)_3 + \left(1 + \frac{1}{2}\rho\right)(k-1)_2\right] x^3 y^2 \\
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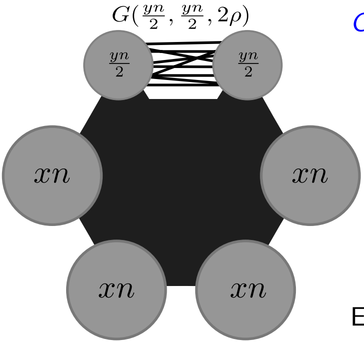

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$$g(x,y,\rho) = (k-1)_2 x^2 + 2(k-1)xy + \rho y^2$$

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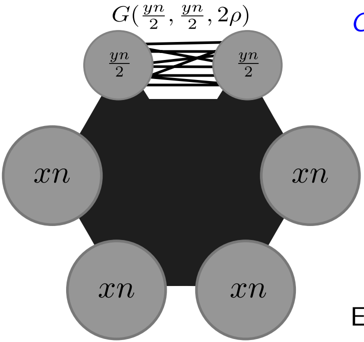

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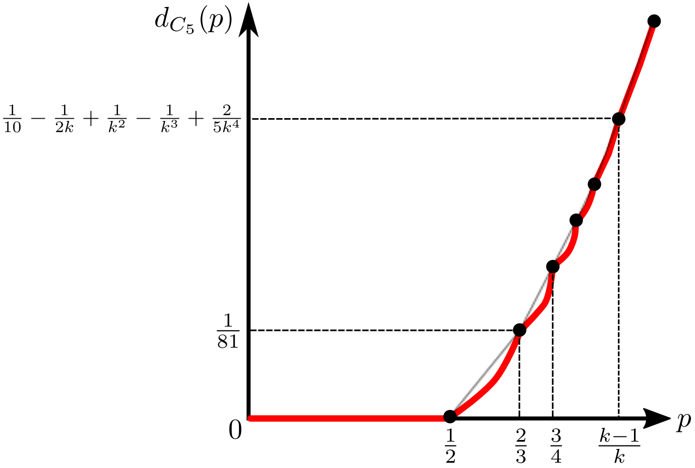
$$\begin{aligned} \texttt{Minimize} \quad & f(x,y,\rho) \\ \texttt{subject to:} \quad & (k-1)x + y = 1, \\ & g(x,y,\rho) = p, \\ & x, y \geq 0. \end{aligned}$$

Numerical results



Flag algebras

Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

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Features:

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

An "easy" example

Theorem (Mantel, 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2$ edges.

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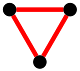
We will use colors for edges and non-edges.

Flag algebras definitions

Let G be a 2-edge-colored complete graph on n vertices.



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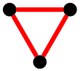

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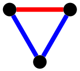

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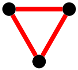

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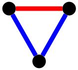

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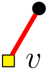

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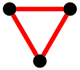

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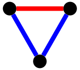

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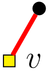

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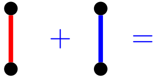

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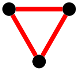

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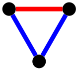

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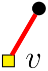

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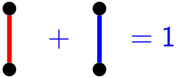

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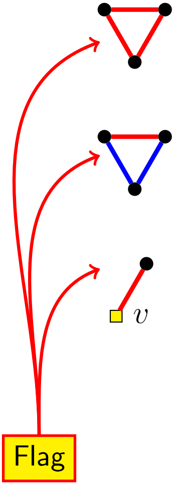

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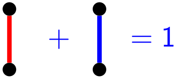

Flag algebras identities

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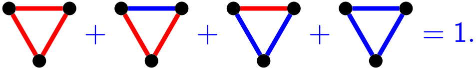
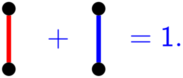


Same as



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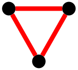
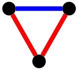
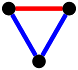
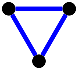
Let G be a 2-edge-colored complete graph on n vertices. Then by the law of total probability

 $= \frac{3}{3}$  $+ \frac{2}{3}$  $+ \frac{1}{3}$  $+ \frac{0}{3}$  .

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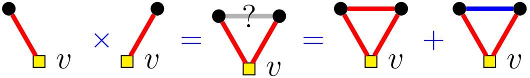

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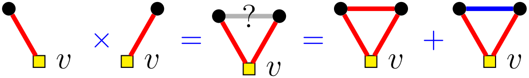

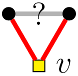

: The probability of choosing two different vertices . . .

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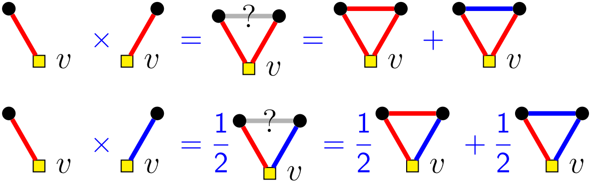

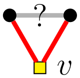

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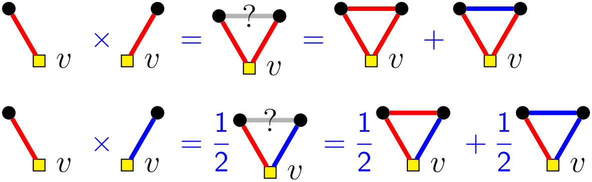

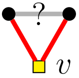

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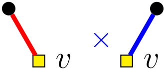
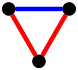
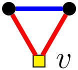


: The probability that choosing two vertices u_1, u_2 other than v gives red vu_1 and blue vu_2.

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices. Then

$\frac{1}{3}$  $= \frac{1}{n} \sum_{v \in V(G)}$  v

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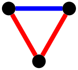
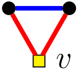
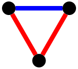
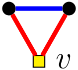
$\frac{1}{3}$  $= \frac{1}{n} \sum_{v\in V(G)}$  v

 $\binom{n}{3} = \sum_{v\in V(G)}$  $v \binom{n-1}{2}$

Flag algebras identities

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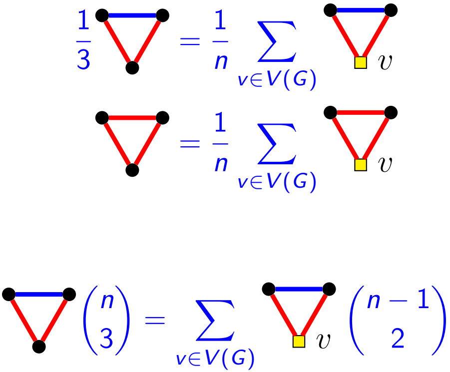

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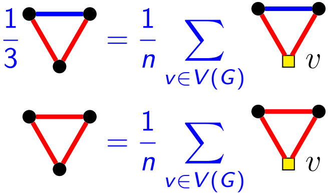
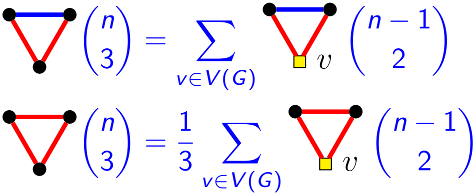
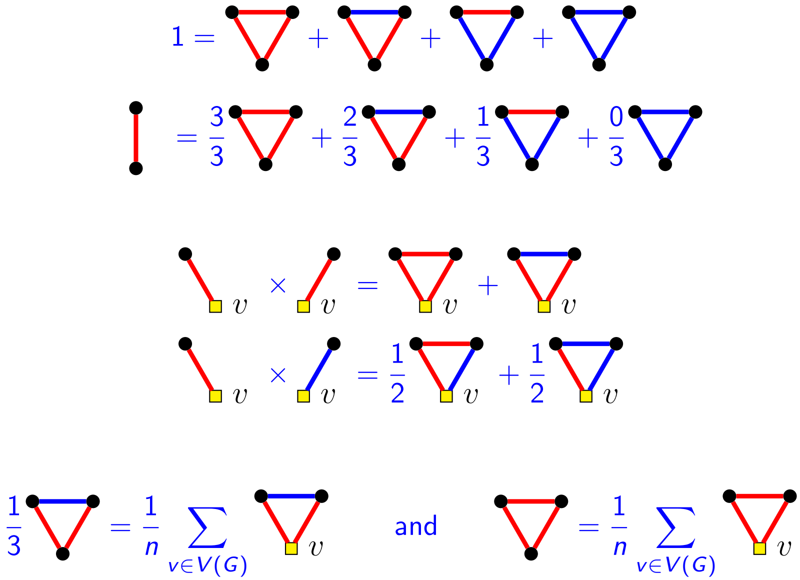




Identities Summary

Let G be a 2-edge-colored complete graph on n vertices. Then



Mantel's theorem - 1st try

Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

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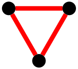
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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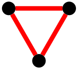
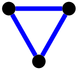
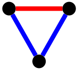
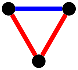
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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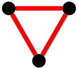
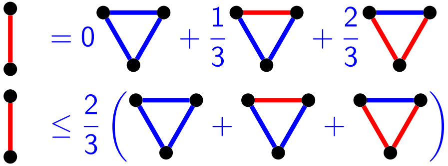
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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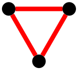
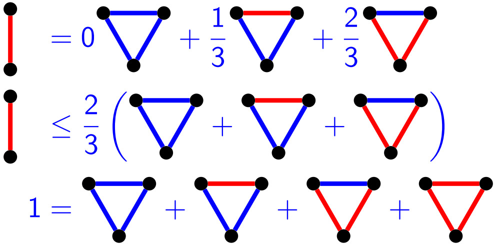
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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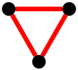
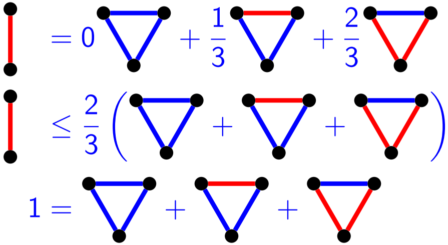
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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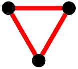
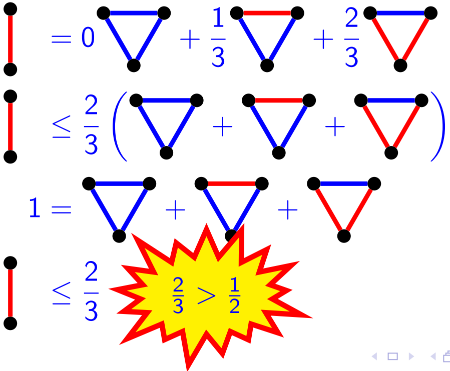
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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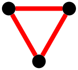
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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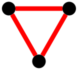
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \Big(1 - 2$  $\Big)^2$

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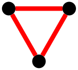
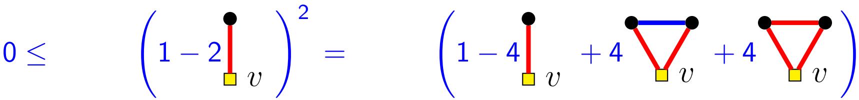
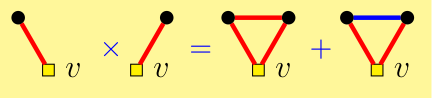
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)





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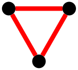
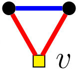
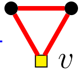
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $_v\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $_v + 4$  $_v + 4$  $_v\Big)$

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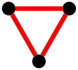
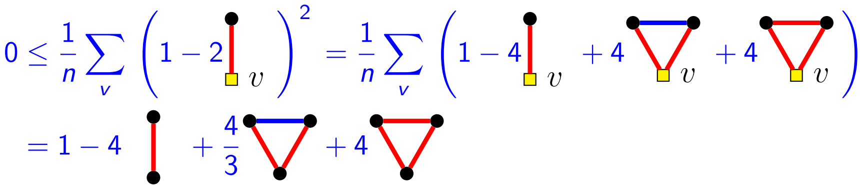
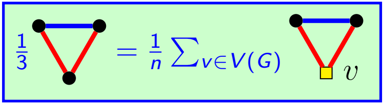
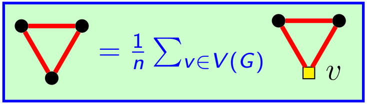
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)







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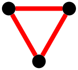
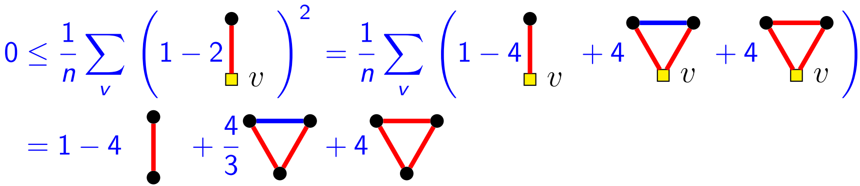
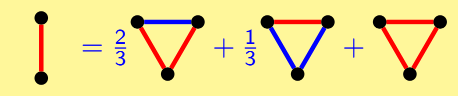
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)





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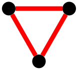
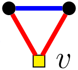
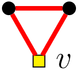
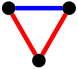
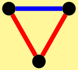
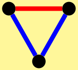
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

$= 1 - 4$  $+ \frac{4}{3}$ 

 $= \frac{2}{3}$  $+ \frac{1}{3}$ 

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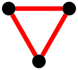
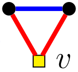
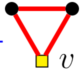
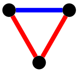
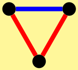
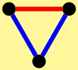
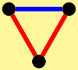
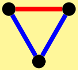
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $+ 4$  $+ 4$  $\Big)$

$= 1 - 4$  $+ \frac{4}{3}$ 

$0 = 2$  $- \frac{4}{3}$  $- \frac{2}{3}$ 

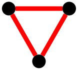
 $= \frac{2}{3}$  $+ \frac{1}{3}$ 

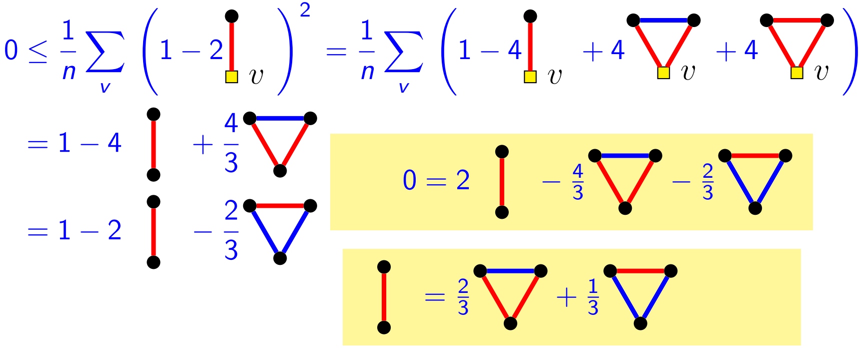
Mantel's Theorem - 2nd try

Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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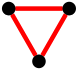
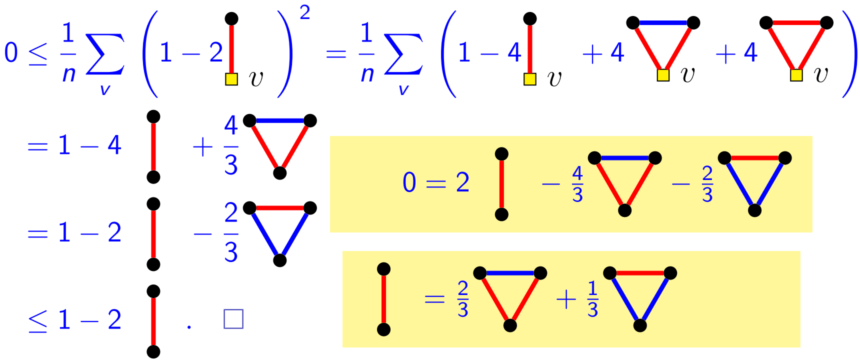
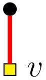
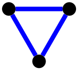
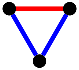
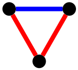
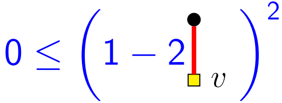
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

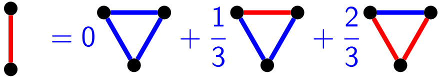


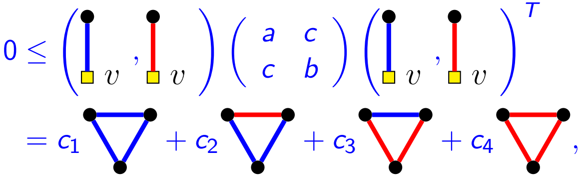
How to guess $0 \le \Big(1 - 2$  $\Big)^2$?

How to guess $0 \le \left(1 - 2$  $\right)^2$?

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

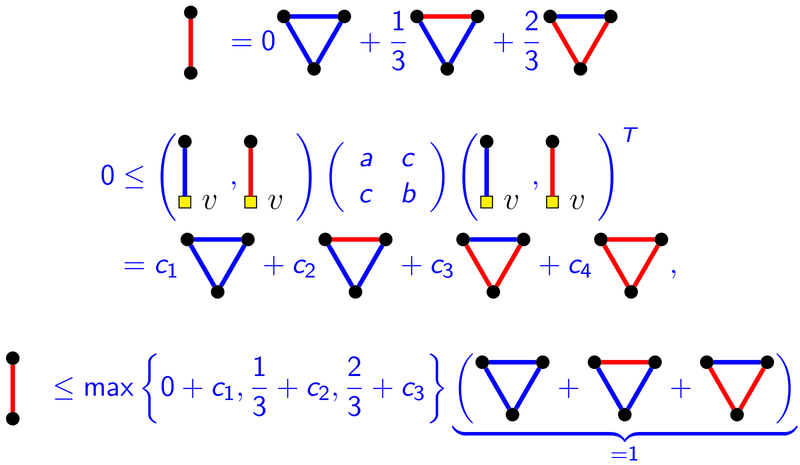
How to guess  ?





$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is a positive semidefinite matrix

How to guess $0 \leq \left(1 - 2\,\cdot_v\right)^2$?

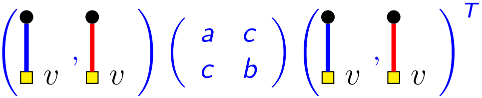


$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is a positive semidefinite matrix, can be optimized.

Candidates for c_1, c_2, c_3

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$ (matrix is positive semidefinite)

Candidates for c_1, c_2, c_3

$0 \leq$ 

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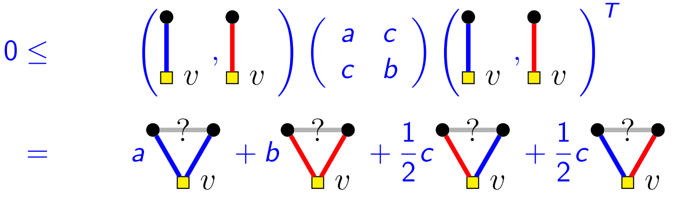

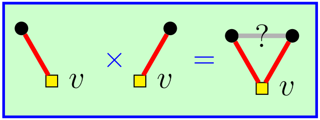

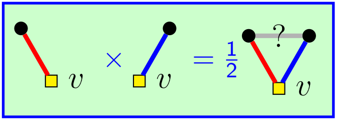

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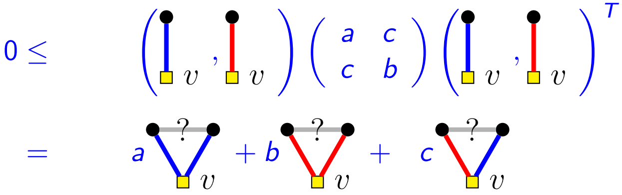

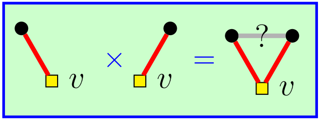

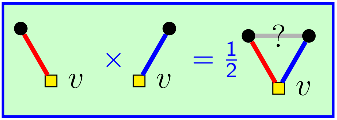

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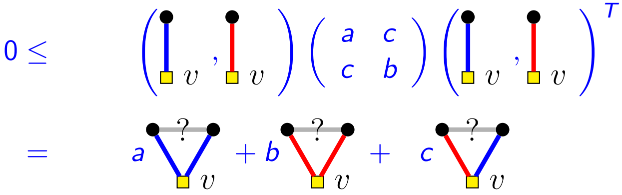

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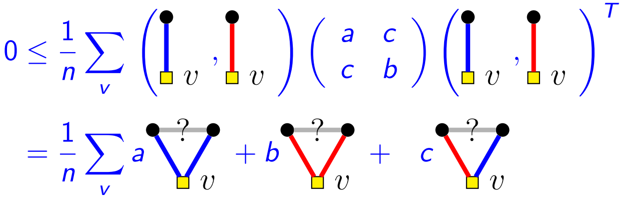

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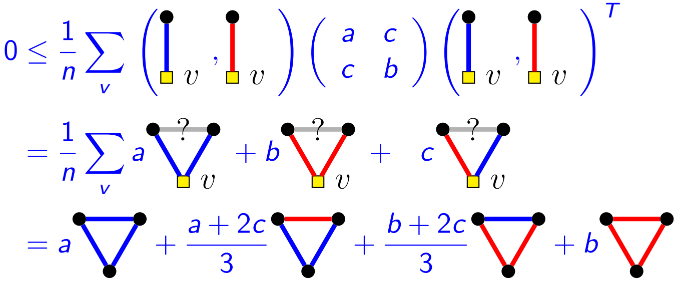
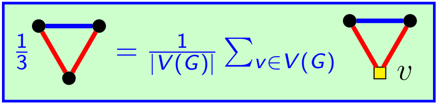
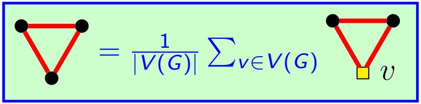
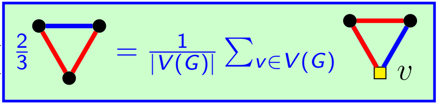






$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (



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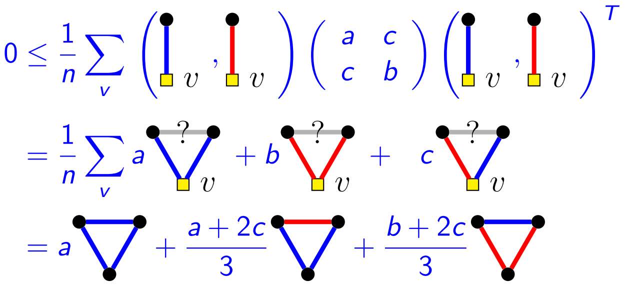

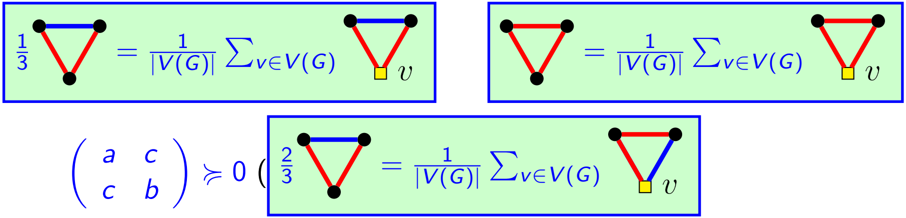

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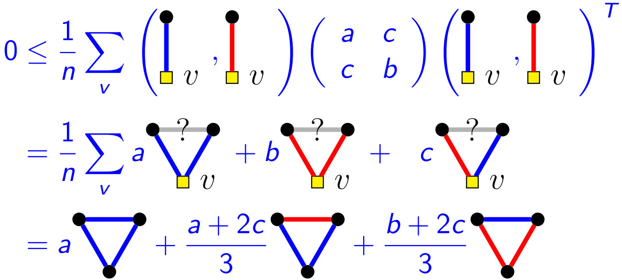
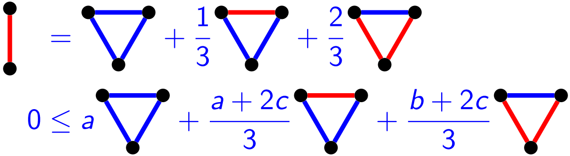


$$c_1 = a,\ c_2 = \frac{a+2c}{3},\ c_3 = \frac{b+2c}{3}$$

$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)

USING c_1, c_2, c_3



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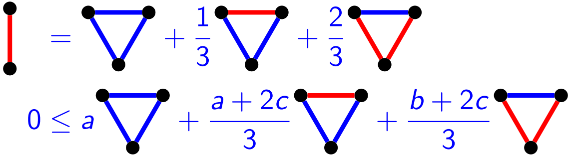

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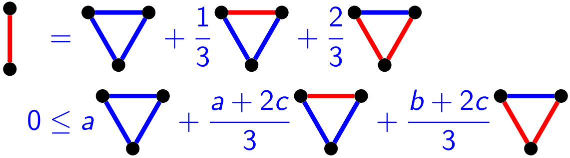

$$\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}.$$

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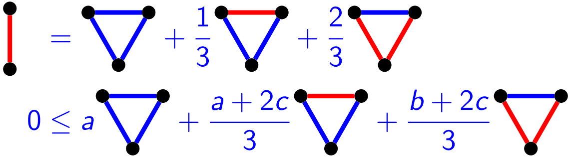
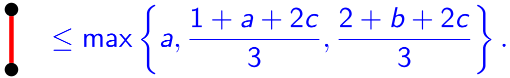
Using c_1, c_2, c_3

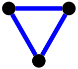
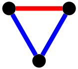
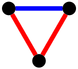


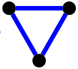
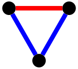
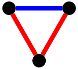


Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Using c_1, c_2, c_3

 $=$  $+\frac{1}{3}$  $+\frac{2}{3}$ 

$0 \leq a$  $+\frac{a+2c}{3}$  $+\frac{b+2c}{3}$ 

 $\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}.$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives

 $\leq \max\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}.$

Optimizing a, b, c

$$\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}$$

$$(SDP)\begin{cases} \texttt{Minimize} & d \\ \texttt{subject to:} & a \leq d \\ & \frac{1+a+2c}{3} \leq d \\ & \frac{2+b+2c}{3} \leq d \\ & \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \end{cases}$$

(SDP) can be solved on computers using CSDP or SDPA.
Rounding may be needed for exact results using SAGE.

Proving the main result

Theorem (Bennett, D, Lidický, 2018)

Let $k \geq 2$ *be an integer and* $p = 1 - \frac{1}{k}$*. Then,*

$$d_{C_5}(p) = \frac{1}{10} - \frac{1}{2k} + \frac{1}{k^2} - \frac{1}{k^3} + \frac{2}{5k^4}.$$

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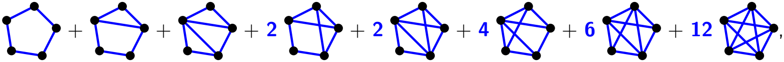
$OPT =$ 

where the coefficient of each graph is the # of copies of C_5 it contains.

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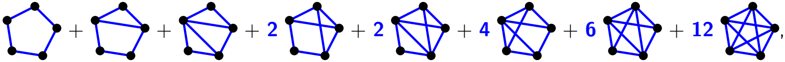
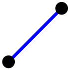
$OPT =$ 

where the coefficient of each graph is the # of copies of C_5 it contains.

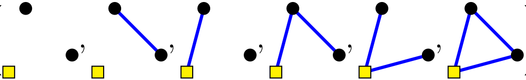
`Minimize` OPT

`subject to:`  $\geq p$

Bounding *OPT*

Let

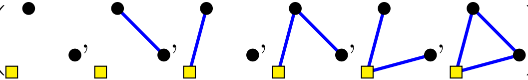
$$X = \Big($$



$$\Big)^T.$$

Bounding *OPT*

Let

$$X = \Big($$  $$\Big)^T.$$

Then, $[\![X^T M X]\!] \geq 0$ for any positive semidefinite matrix $M = M_{6\times 6}$ and it is a linear combination of all graphs on 5 vertices (denoted by $\mathcal{F}_5$).

Bounding *OPT*

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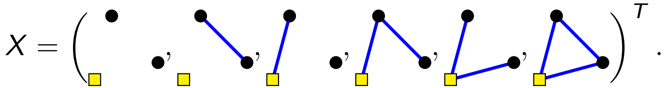

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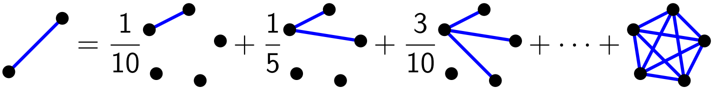
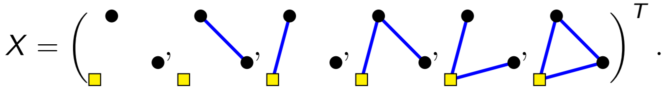


Bounding *OPT*

Let



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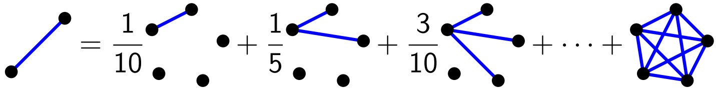
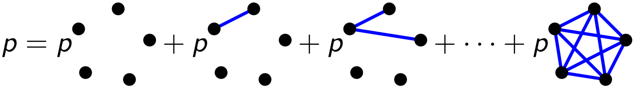
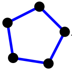
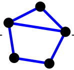
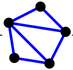
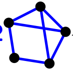
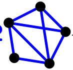
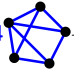
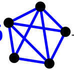
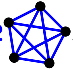


and



Bounding *OPT*

Since *OPT* =  +  +  + **2**  + **2**  + **4**  + **6**  + **12** ,

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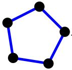
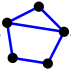
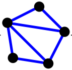
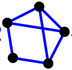
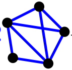
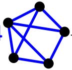
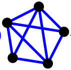
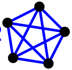
Since $OPT =$  $+$  $+$  $+\mathbf{2}$  $+\mathbf{2}$  $+\mathbf{4}$  $+\mathbf{6}$  $+\mathbf{12}$ ,

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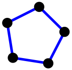
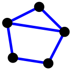
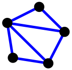
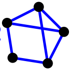
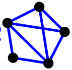
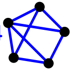
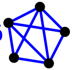
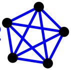
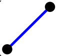
$OPT \geq$  $+$  $+$  $+\ \mathbf{2}$  $+\ \mathbf{2}$  $+\ \mathbf{4}$  $+\ \mathbf{6}$  $+\ \mathbf{12}$ 

$- \alpha \Big($  $- p \Big) - [\![X^T M X]\!]$

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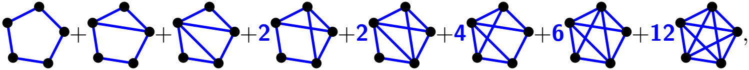
Since $OPT =$ 

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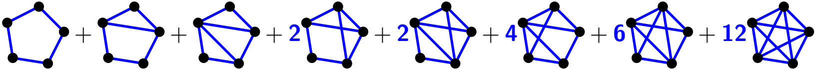
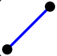
$OPT \geq$ 

$- \alpha \Big($  $- p \Big) - [\![X^T M X]\!]$

$$= \sum_{F \in \mathcal{F}_5} c_F \cdot F \geq \min_{F \in \mathcal{F}_5} c_F \cdot \sum_{F \in \mathcal{F}_5} F = \min_{F \in \mathcal{F}_5} c_F.$$

Clearly, each $c_F = c_F(M, \alpha)$.

Optimizing coefficients c_F

In order to define matrix M we define first two matrices A and B:

$$A = \begin{pmatrix} 32k^2 - 96k + 96 & 0 & 4k^2 - 16k \\ 0 & 10k^4 - 30k^3 - 8k^2 + 96k - 96 & -10k^4 + 35k^3 - 4k^2 - 80k + 96 \\ 4k^2 - 16k & -10k^4 + 35k^3 - 4k^2 - 80k + 96 & 10k^4 - 40k^3 + 24k^2 + 64k - 96 \end{pmatrix}$$

and

$$B = \begin{pmatrix} k-1 & 1 & k-2 & 0 & k-3 & -1 \\ 0 & 2 & k-2 & 0 & 2k-4 & -2 \\ 0 & 0 & k-1 & -1 & 2k-2 & -2 \end{pmatrix}.$$

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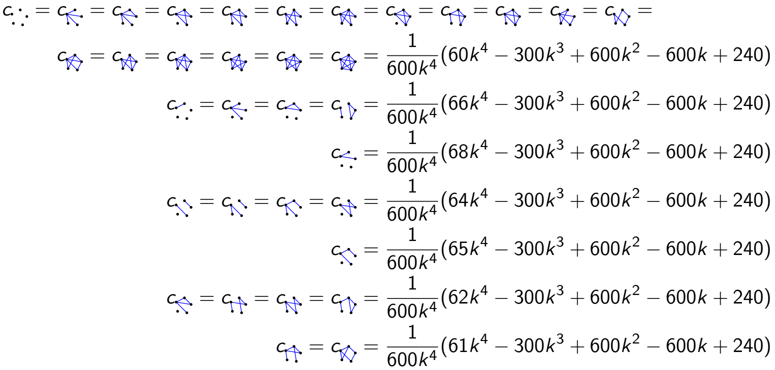
Set

$$\alpha = \frac{1}{600k^4} \left(30k^3 - 120k^2 + 180k - 96\right).$$

Optimizing coefficients c_F



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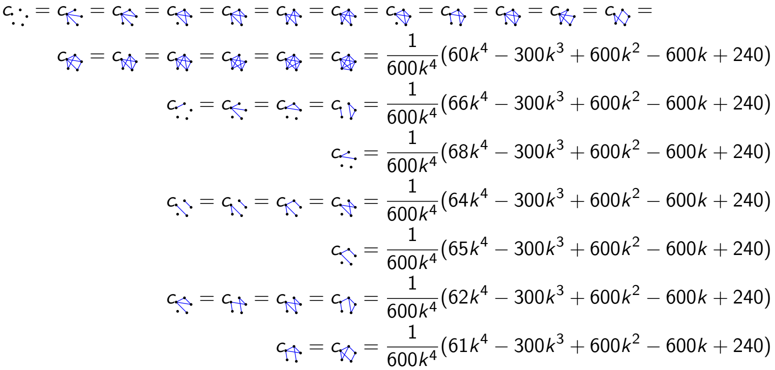

Thus,

$$OPT \geq \min_{F \in \mathcal{F}_5} c_F = \frac{1}{600k^4}(60k^4 - 300k^3 + 600k^2 - 600k + 240) = \frac{1}{10} - \frac{1}{2k} + \frac{1}{k^2} - \frac{1}{k^3} + \frac{2}{5k^4}.$$

Stability

Once again,

$$12 - \frac{60}{k} + \frac{120}{k^2} - \frac{120}{k^3} + \frac{48}{k^4} \geq OPT \geq \sum_{F \in \mathcal{F}_5} c_F \cdot F \geq \min_{F \in \mathcal{F}_5} c_F \cdot \sum_{F \in \mathcal{F}_5} F$$

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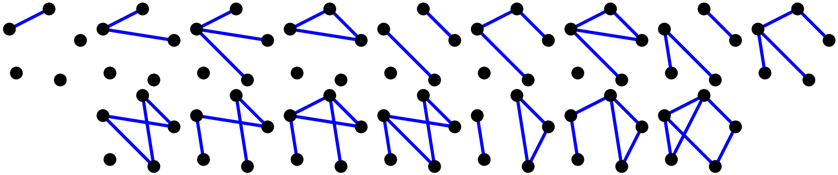
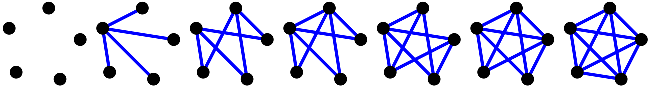
Thus, if $c_F > \min_{H \in \mathcal{F}_5} c_H$, then the density of F must be zero.

And so the following graphs have zero density in the limit:



Stability

As a matter of fact for $2 \leq k \leq 73$ one can show by using flags with more labeled vertices that the only possible graphs with nonzero density must belong to the following list $\mathcal{L}$:



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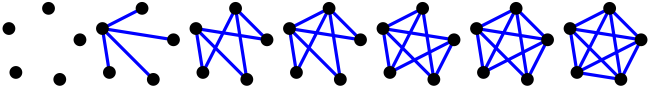

Apply to G *the induced graph removal lemma* to eliminate all induced subgraphs that are in not in $\mathcal{L}$. Call this new graph G'. Observe that G' has no induced copy of  and therefore is a complete partite graph.

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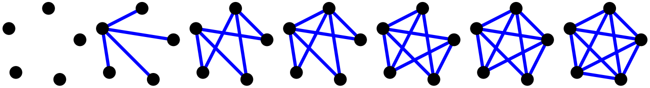

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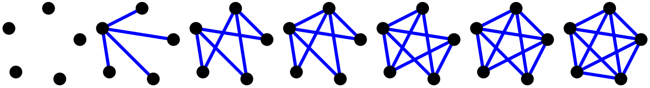

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Simple but tedious calculations show that G' is k-partite.

And so the graphs with density $p = 1 - \frac{1}{k}$ that minimize the number of copies of C_5 are "close" to the Turán graph.

Further directions

Determine $d_{C_5}(p)$ for $p \neq 1 - \frac{1}{k}$.

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General stability results.

Thanks!