The sandwich conjecture of random regular graphs and more

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(together with P. Gao and B.D. McKay)

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Introduction

Random graphs

The parameter $\boldsymbol{n}$ is the number of vertices. All graphs are labelled.

- $\boldsymbol{\mathcal{G}(n,p)}$ **model**: every pair of vertices is connected in the graph with probability $\boldsymbol{p}$ independently from every other edge.
- $\boldsymbol{\mathcal{G}(n,m)}$ **model**: we take a uniform random element of the set of graphs on $\boldsymbol{n}$ vertices with $\boldsymbol{m}$ edges.
- $\boldsymbol{\mathcal{R}(n,d)}$ **model**: we take a uniform random element of the set of $\boldsymbol{d}$-regular graphs on $\boldsymbol{n}$ vertices (we always assume $\boldsymbol{dn}$ is even).

The sandwich conjecture

Conjecture (Kim and Vu, 2004)

For $d \gg \log n$, there is a random triple (G_1, R, G_2) of graphs on n vertices which marginal distributions are

$$G_1 \sim \mathcal{G}(n, p_1), \qquad R \sim \mathcal{R}(n, d), \qquad G_2 \sim \mathcal{G}(n, p_2),$$

for some $p_1 = \frac{d}{n}(1 - o(1))$ and $p_2 = \frac{d}{n}(1 + o(1))$, and

$$\Pr(G_1 \subseteq R \subseteq G_2) = 1 - o(1).$$

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Kim and Vu managed to prove the sandwich conjecture for the range $\log n \ll d \leq n^{1/3 - o(1)}$ with a defect in one side: $\mathcal{R}(n, d)$ is not completely contained in $\mathcal{G}(n, p_2)$.

Recent progress towards the sandwich conjecture

Dudek, Frieze, Ruciński, and Šileikis [J. Comb. Theory B, 2017] showed if $\boldsymbol{d = o(n)}$ then $\boldsymbol{\mathcal{G}(n, p_1) \subseteq \mathcal{R}(n, d)}$ a.a.s. with $\boldsymbol{p_1 = (1 - o(1))\frac{d}{n}}$.

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Theorem (Gao, I., McKay)

Let ε be any positive constant. Then the following holds a.a.s.

(i) For $d \geq n^{2/3+\varepsilon}$ the sandwich conjecture holds, i.e.

$$\mathcal{G}(n, p_1) \subseteq \mathcal{R}(n, d) \subseteq \mathcal{G}(n, p_2)$$

with $p_1 = (1 - o(1))\frac{d}{n}$ and $p_2 = (1 + o(1))\frac{d}{n}$.

(ii) If $d \geq n^{1/2}$ then $\mathcal{R}(n, d) \subseteq \mathcal{G}(n, p_2)$ with $p_2 = \varepsilon\frac{d}{n}\log n$.

(iii) If $d \leq n^{1/2}$ then $\mathcal{R}(n, d) \subseteq \mathcal{G}(n, p_2)$ with $p_2 = \varepsilon n^{-1/2}\log n$.

Coupling procedure

Another way to generate $\mathcal{G}(n,p)$

Procedure $M(n,m)$.

1. Take $M := \emptyset$.
2. Repeat m times: take jk uniformly at random from K_n and add it to M (in case the edge jk was not in M yet).
3. Return M.

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Let $M_\xi(n, m)$ be the random graph defined similarly to $M(n, m)$ but with some rejection probability ξ at Step 2. Then,

$$M_\xi(n, D) \sim \mathcal{G}(n, p_\xi) \text{ with } p_\xi = 1 - e^{-\lambda(1-\xi)/\binom{n}{2}}.$$

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Kim and Vu relied on the algorithm of [Steger and Wormald, Combin.Probab. Comput., 1999] and the asymptotic formula for the number of d-regular graphs.

Coupling $\mathcal{G}(n, p) \subseteq \mathcal{R}(n, d)$.

Procedure $R(n, d)$.

1. Take $R := \emptyset$.
2. Repeat until R is d-regular: take jk uniformly at random from K_n and add it to R with probability

$$\frac{\mathbf{Pr}(jk \in \mathcal{R}(n,d) \mid R \subset \mathcal{R}(n,d))}{\mathbf{max}_{jk \notin R} \mathbf{Pr}(jk \in \mathcal{R}(n,d) \mid R \subset \mathcal{R}(n,d))} \tag{1}$$

   (in case the edge jk was not in R yet).
3. Return R.

Idea: to achieve $M_\xi(n, D) \subseteq R(n, d)$ we only need to show that a.a.s. (1) is bounded below by $1 - \xi$ for the first D iterations of Step 2.

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Let $S \sim \mathcal{G}(n, q)$. Take a t-factor $T \subseteq S$ uniformly at random.

Toy problem

For which values of q and t we can show a.a.s.

$$\mathbf{Pr}_S(uv \in T) = (1 + o(1))\frac{t}{qn}$$

simultaneusly for all edges $uv \in S$?

For the coupling procedure we need:

- q ranges from 1 to $1 - \frac{d}{n}$
- t ranges from d to 0.

Why is one side more difficult than another?

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Let T is a uniform random t-factor of $S \sim \mathcal{G}(n, q)$. We need

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It is fairly easy to resolve the toy problem for $d = o(n)$ which gives us $\mathcal{G}(n, p_1) \subseteq \mathcal{R}(n, d)$ with $p_1 = \frac{d}{n}(1 - o(1))$, see [Dudek et al., 2017].

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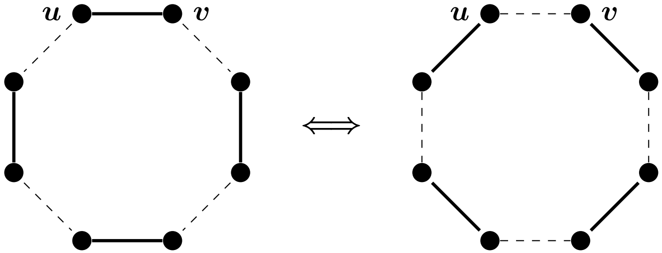
However to show the containment $\mathcal{R}(n,d) \subseteq \mathcal{G}(n,p_2)$ is equivalent to $\mathcal{G}(n, 1-p_2) \subseteq \mathcal{R}(n,d')$ with $d' = n - 1 - d = n - o(n)$. So we need to resolve the toy problem with $q = o(1)$ for that.

Two key ideas

Switchings

Fix $\boldsymbol{S}$ and $\boldsymbol{T}$. Mark edges in $\boldsymbol{T}$ by — and edges in $\boldsymbol{S}-\boldsymbol{T}$ by - - -.



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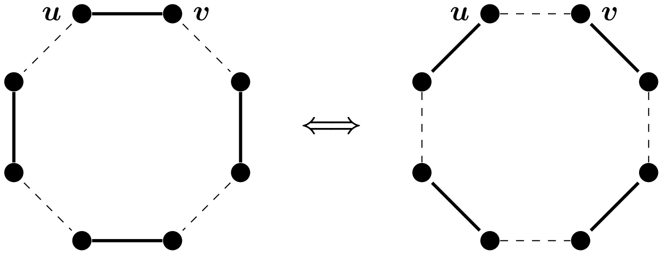

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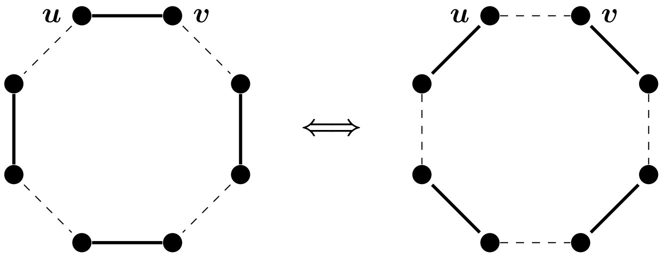

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This works for $q \geq \varepsilon n^{-1/2} \log n$ and $t = o(qn)$.

Complex-analytic approach

The probability can be expressed as a ratio of two integrals:

$$\mathbf{Pr}_S(uv \in T) = \frac{\oint \dots \oint \frac{\prod_{jk\in S-uv}(1+z_j z_k)}{z_1^{t+1}\dots z_n^{t+1}/z_u z_v} dz_1 \dots dz_n}{\oint \dots \oint \frac{\prod_{jk\in S}(1+z_j z_k)}{z_1^{t+1}\dots z_n^{t+1}} dz_1 \dots dz_n} = \cdots$$

Then, we estimate these multidimensional complex integrals using the machinery of [I., McKay, Random Struct. Algor., 2017] and get that

$$\cdots = (1+o(1)) \frac{\frac{t}{qn} \cdot \frac{1}{(2\pi)^{n/2}|Q_S|} e^{\mathbb{E}g(X) - \frac{1}{2}\mathbb{E}h(X)^2 + o(1)}}{\frac{1}{(2\pi)^{n/2}|Q_{S-uv}|} e^{\mathbb{E}\tilde{g}(\tilde{X}) - \frac{1}{2}\mathbb{E}\tilde{h}(\tilde{X})^2 + o(1)}} = (1+o(1))\frac{t}{qn}.$$

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This works for $q \geq n^{-1/3+\varepsilon}$ and $\min\{t, qn-t\} \gg qn/\log n$.

...AND MORE

More sandwiches

1) Our result actually covers random graphs with given degree sequence $(d_1, \ldots, d_n)$ that $d_j = d(1 + o(1))$.

2) Similar sandwiching results holds for the model G_p and random subgraph of G with given degrees (chosen uniformly).

3) There are immediate corollaries of the form $\mathcal{R}(n, d_1) \subseteq \mathcal{R}(n, d_2)$.

Thank you for your attention!