On the Reconstruction of Graphs of Connectivity 2

Dmitri Karpov

St.Petersburg Department of V. A. Steklov Institute of Mathematics of the Russian Academy of Sciences, St.Petersburg, Russia

Workshop on Graphs, Networks and their Applications
16.05.2018

Graph reconstruction conjecture

For a graph G with $V(G) = \{v_1, \dots, v_n\}$, let and $\mathcal{D}(G)$ be the collection of graphs $G - v_1, \dots, G - v_n$.

Conjecture

(P. Kelly, 1957 and S. Ulam, 1960.)
Let G and H be graphs with $v(G) = v(H) \geq 3$, such that $\mathcal{D}(G) = \mathcal{D}(H)$. Then these graphs are isomorphic.

Reconstructible graphs and parameters

Consider $\mathcal{D}(G)$, where $v(G) \geq 3$.

Definition

A graph G is *reconstructible*, if it is uniquely determined by $\mathcal{D}(G)$ (up to isomorphism).
A graph parameter is *reconstructible*, if it is uniquely determined by $\mathcal{D}(G)$.
A graph property is *reconstructible*, if it can be checked by $\mathcal{D}(G)$ whether G has this property or not.

The Graph Reconstruction Conjecture claims, that all graphs on at least 3 vertices are reconstructible.

What is trivially reconstructible?

- The number of edges $e(G)$ and the collection of vertex degrees
- The vertex connectivity $\kappa(G)$, i.e. the size of the least vertex cutset
- Disconnected graphs

Reconstructible classes of graphs

- Disconnected graphs
- Trees (P. Kelly, 1957)
- Graphs of connectivity 1 without pendant vertices, i.e. graphs G with $\kappa(G) = 1$ and $\delta(G) \geq 2$ (J. A. Bondy, 1966)
- All graphs of connectivity 1 (Y. Yongzhi, 1988).

Tree of blocks and cutpoints

Let G be a connected graph.

A vertex $a \in V(G)$ is a *cutpoint*, if the graph $G - a$ is disconnected.

A *block* of the graph G is a maximal up to inclusion subgraph, having no cutpoints.

The structure of mutual disposition of blocks and cutpoints of a connected graph can be described by the *tree of blocks and cutpoints* $B(G)$.

Vertices of the first partition are all cutpoints $a_1, \dots, a_n$ of the graph G, vertices of the second partition are all blocks $B_1, \dots, B_m$ of the graph G. Vertices a_i and B_j are adjacent if and only if $a_i \in V(B_j)$.

Tree of blocks and cutpoints

It is easy to prove, that $B(G)$ is a tree, all leaves of which correspond to blocks (which are called *pendant blocks*).

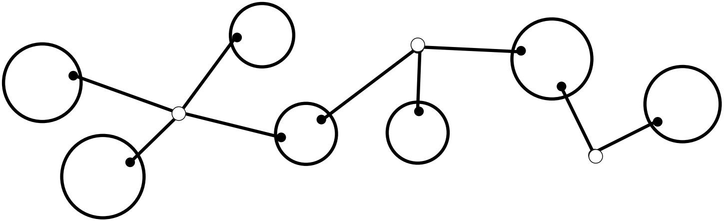

Figure: A tree of blocks and cutpoints

Reconstruction of graphs of connectivity 1

For a graph G with $\kappa(G) = 1$ and $\delta(G) \geq 2$ the following parameters are reconstructible:

- The minimal size of a pendant block;
- The tree of blocks and cutpoints
- The collection of pendant blocks with their only cutpoints marked.

This helps to reconstruct graphs of connectivity 1:

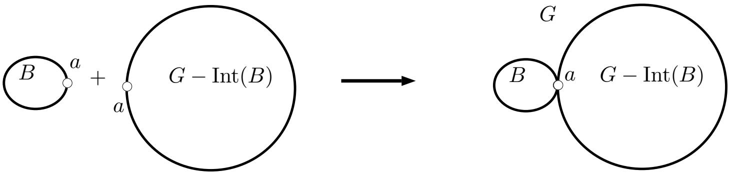

Figure: Gluing by a cutpoint

Semi-reconstruction of graphs of connectivity 2

Theorem 1

(D.K., 2018.) *Let* G *be a graph with* $\kappa(G) = 2$ *and* $\delta(G) \geq 3$*. Having* $\mathcal{D}(G)$*, we can find a pair of graphs* G_1, G_2*, such that* $G \in \{G_1, G_2\}$*.*

The proof of Theorem 1 is like to Bondy's one with *block tree of a 2-connected graph* instead of the tree of blocks and cutpoints.

Block tree of graphs of connectivity 2

Let G be a graph with $\kappa(G) = 2$.

Denote by $\mathfrak{R}_2(G)$ the set of all its 2-vertex cutsets

Let $A \subset V(G)$, $T \in \mathfrak{R}_2(G)$. The cutset T *splits* A, if A is disconnected in $G - T$.

A cutset $S \in \mathfrak{R}_2(G)$ is *single*, if no cutset of $\mathfrak{R}_2(G)$ splits S.

A *part* of G is a maximal up to inclusion vertex set such that no single cutset splits A.

Denote by $\mathrm{Part}(G)$ the set of all parts of G.

Block tree of graphs of connectivity 2

Let G be a graph with $\kappa(G) = 2$.

The block tree $\mathrm{BT}(G)$ is a graph with bipartition $(\mathfrak{O}(G), \mathrm{Part}(G))$.
Vertices $S \in \mathfrak{O}(G)$ and $A \in \mathrm{Part}(G)$ are adjacent if and only if $S \subset A$.

Then $\mathrm{BT}(G)$ is a tree, all leaves of which correspond to *pendant parts* of $\mathrm{Part}(G)$.

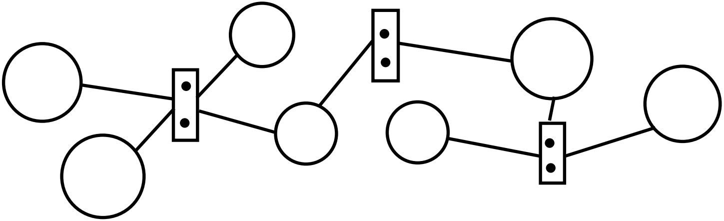

Figure: A block tree of a graph of connectivity 2

Semi-reconstruction of graphs of connectivity 2

For a graph G with $\kappa(G) = 2$ and $\delta(G) \geq 3$ the following parameters are reconstructible:

- The minimal size of a pendant part;
- The tree $\mathrm{BT}(G)$;
- The collection of subgraphs induced on pendant parts with their only single cutset marked.

Semi-reconstruction of graphs of connectivity 2

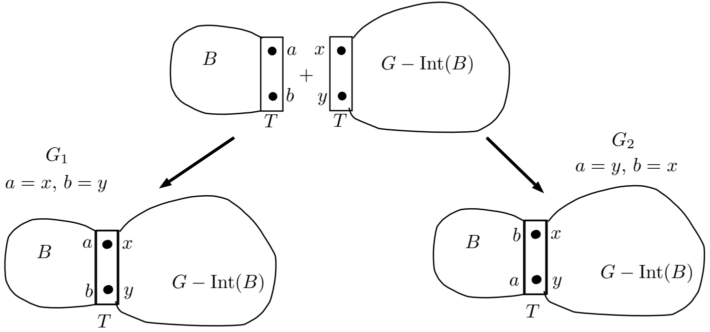

Figure: Two ways of gluing by a 2-vertex cutset

A result on real reconstruction

But how can we decide what graph of the pair G_1, G_2 provided by Theorem 1 is our graph G? In what cases this can be done?

Theorem 2

(D.K., 2018.) *Let G be a 2-connected graph with $\delta(G) \geq 3$. Let there exists a cutset $T \subset V(G)$ such that $|T| = 2$ the graph $G - T$ has at least 3 connected components. Then G is reconstructible (i.e., any graph H with $\mathcal{D}(H) = \mathcal{D}(G)$ is isomorphic to G).*

Note, that the property of having such a cutset T is easily reconstructible.