Features of DP-coloring of graphs and multigraphs

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joint work with A. Bernshteyn, S. P. Pron and X. Zhu

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A list L for a graph G is a map $L : V(G) \to \mathrm{Pow}(\mathbb{Z}_{>0})$ that assigns to each vertex $v \in V(G)$ a set $L(v) \subseteq \mathbb{Z}_{>0}$.
An L-coloring of G is a mapping $f : V(G) \to \mathbb{Z}_{>0}$ such that $f(v) \in L(v)$ for each $v \in V(G)$ and $f(v) \neq f(u)$ whenever $vu \in E(G)$.
The list chromatic number, $\chi_\ell(G)$, is the minimum k such that G has an L-coloring for each L satisfying $|L(v)| = k$ for every $v \in V(G)$.

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The plan did not work as planned, but the new notion (introduced also by Erdős, Rubin and Taylor) turned out to be valuable and interesting. Some properties of it are very close to those of the ordinary coloring, and some are quite different.

One well-known application of list coloring is the Fleischner-Stiebitz proof of the cycle-plus-triangles problem by Erdős.

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Theorem 1 [Gallai, 1963] If $k \geq 3$ and G is a k-critical graph, then the subgraph of G induced by the vertices of degree $k-1$ is a Gallai forest.

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A list L for a graph G is a degree list if $|L(v)| \geq \deg_G(v)$ for all $v \in V(G)$.

Theorem 2 [Borodin, 1976; Erdős–Rubin–Taylor, 1979] Let G be a connected graph and let L be a degree list assignment for G. If G is not L-colorable, then G is a Gallai tree; furthermore, $|L(u)| = \deg_G(u)$ for all $u \in V(G)$ and if $u, v \in V(G)$ are two adjacent non-cut vertices, then $L(u) = L(v)$.

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Theorem 3 [Plesnevič and Vizing, 1965] A graph G has a k-coloring if and only if the Cartesian product $G \square K_k$ contains an independent set of size $|V(G)|$, i.e., $\alpha(G \square K_k) = |V(G)|$.

Vizing's Theorem (1976)

Given a list L for G, the vertex set of the auxiliary graph $H = H(G, L)$ is $\{(v, c) : v \in V(G) \text{ and } c \in L(v)\}$, and two distinct vertices (v, c) and (v', c') are adjacent in H if and only if either $c = c'$ and $vv' \in E(G)$, or $v = v'$.

Since $V(H)$ is covered by $|V(G)|$ cliques, $\alpha(H) \leq |V(G)|$. If H has an independent set I with $|I| = |V(G)|$, then, for each $v \in V(G)$, there is a unique $c \in L(v)$ such that $(v, c) \in I$. And the same color c is not chosen for any two adjacent vertices. So the map $f : V(G) \to \mathbb{Z}_{>0}$ defined by $(v, f(v)) \in I$ is an L-coloring of G.

Also, if G has an L-coloring f, then the set $\{(v, f(v)) : v \in V(G)\}$ is an independent set of size $|V(G)|$ in H.

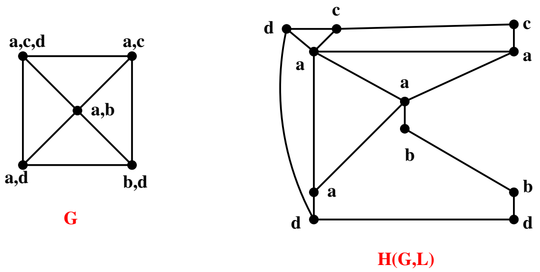

Figure: A graph G with a list L and a cover for (G, L).

Definition

Let G be a graph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$, satisfying the following:

1. For each $v \in V(G)$, $H[L(v)]$ is a complete graph.
2. For each $uv \in E(G)$, the edges between $L(u)$ and $L(v)$ form a matching (possibly empty).
3. For each distinct $u, v \in V(G)$ with $uv \notin E(G)$, no edges of H connect $L(u)$ and $L(v)$.

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Let G be a graph and (L, H) be a cover of G. An *(L, H)-coloring* of G is an independent set $I \subseteq V(H)$ of size $|V(G)|$. G is *(L, H)-colorable* if it admits an (L, H)-coloring.

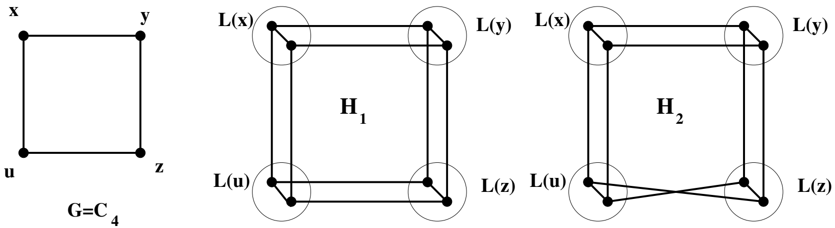

Figure: Graph C_4 and two covers of it such that C_4 is (L, H_1)-colorable but not (L, H_2)-colorable.

The *DP-chromatic number*, $\chi_{DP}(G)$, is the minimum k such that G is (L, H)-colorable for each choice of (L, H) with $|L(v)| \geq k$ for all $v \in V(G)$.

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6. $\chi_{DP}(G) \leq C\frac{d}{\ln d}$ for every triangle-free G with maximum degree d. (A. B.)

Multigraphs

Let G be a multigraph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$ such that

1. For each $v \in V(G)$, $H[L(v)]$ is a complete graph.
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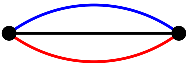

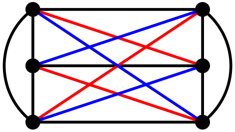

Figure: $\chi_{DP}(K_2^3) = 4$.

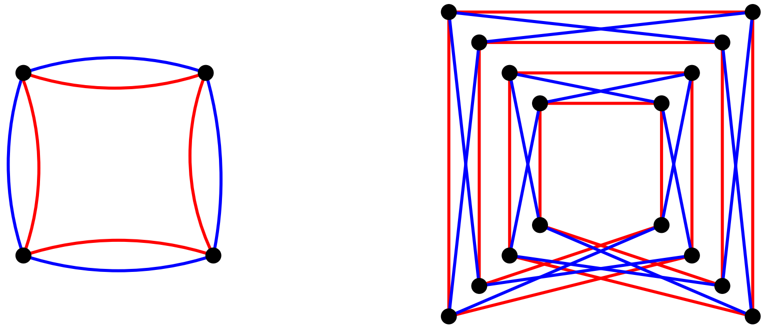

Figure: $\chi_{DP}(C_4^k) = 2k + 1$.

Theorem 4 [Bernshteyn-Pron-A.K., 2016] Let G be a connected multigraph. Then G is not DP-degree-colorable if and only if each block of G is one of the graphs K_n^k, C_n^k for some n and k.

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Corollary 5 [B-P-K] Let $k \geq 4$ and let G be a DP-k-critical graph distinct from K_k. Then

$$2|E(G)| \geq \left(k - 1 + \frac{k-3}{k^2-3}\right) n.$$

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Theorem 6 [Dirac, 1957] Let $k \geq 4$ and let G be a k-critical graph distinct from K_k. Set $n := |V(G)|$ and $m := |E(G)|$. Then

$$2m \geq (k-1)n + k - 3.$$

For $k \geq 4$, a graph G is *k-Dirac* if $V(G)$ can be partitioned into three subsets V_1, V_2, V_3 so that

(a) $|V_1| = k - 1$, $|V_2| = k - 2$, $|V_3| = 2$;

(b) the graphs $G[V_1]$ and $G[V_2]$ are complete;

(c) each $y_i \in V_1$ is adjacent to exactly one $z_j \in V_3$, and each $z_j \in V_3$ has a neighbor in V_1;

(d) each $x_i \in V_2$ is adjacent to both $z_j \in V_3$; and

(e) G has no other edges.

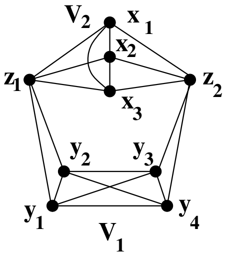

Figure: A 5-Dirac graph.

Let $\mathcal{D}_k$ denote the family of all k-Dirac graphs.

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Theorem 8 [A.K.-Stiebitz, 2002] Let $k \geq 4$ and let L be a list assignment for G such that G is L-critical and $|L(u)| = k - 1$ for all $u \in V(G)$. Suppose that G does not contain a clique of size k. Set $n := |V(G)|$ and $m := |E(G)|$. Then

$$2m \geq (k-1)n + k - 3.$$

Question [A.K.-Stiebitz, 2002] Does Theorem 7 hold for list coloring?

Theorem 9 [B-K] Let $k \geq 4$, G be a graph and let (L, H) be a cover of G such that G is (L, H)-critical and $|L(u)| = k - 1$ for all $u \in V(G)$. Suppose that G does not contain a clique of size k. Set $n := |V(G)|$ and $m := |E(G)|$. If $G \notin \mathcal{D}_k$, then

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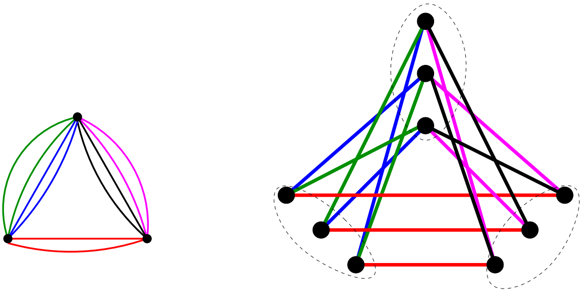

Figure: A DP-7-critical multigraph.

When $\chi_{DP}(G) = \chi(G)$?

The result below was a conjecture by Ohba:

Theorem 10 [Noel–Reed–Wu, 2015] Let G be an n-vertex graph with $\chi(G) \geq (n-1)/2$. Then $\chi_\ell(G) = \chi(G)$.

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Let $\mathcal{J}(G, s)$ denote the *join* of G and a copy of K_s, obtained from G by adding s new vertices that are adjacent to every vertex in $V(G)$ and to each other.

By definition, for all G and s, $\chi(\mathcal{J}(G,s)) = \chi(G) + s$, and $\chi_\ell(\mathcal{J}(G,s)) \leq \chi_\ell(G) + s$.
Theorem 10 yields: for each G and every $s \geq |V(G)| - 2\chi(G) - 1$,

$$\chi_\ell(\mathcal{J}(G,s)) = \chi(\mathcal{J}(G,s)),$$

even if $\chi_\ell(G)$ is much larger than $\chi(G)$.

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$$Z_{DP}(G) := \min\{s \in N : \chi_{DP}(\mathcal{J}(G,s)) = \chi(\mathcal{J}(G,s))\},$$

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and conjectured that for all graphs G, $Z_{DP}(G)$ is finite.

Theorem 11 [Bernshteyn–A.K.–Zhu, 2017] Let G be a graph with n vertices, m edges, and chromatic number k. Then $Z_{DP}(G) \leq 3m$. Moreover, if $\delta(G) \geq k - 1$, then

$$Z_{DP}(G) \leq 3m - \frac{3}{2}(k-1)n.$$

In particular, for all n, $Z_{DP}(n) \leq 3n^2/2$.

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In contrast with the linear upper bound on $Z_\ell(n)$,

Theorem 12 [B–K–Z, 2017] For all n, $Z_{DP}(n) \geq n^2/4 - O(n)$. In particular, if $r(n)$ is the minimum r s. t. for every n-vertex graph G with $\chi(G) \geq r$, we have $\chi_{DP}(G) = \chi(G)$, then

$$n - r(n) = \Theta(\sqrt{n}).$$

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Theorem 16 [B–K, 2017+] If $d \geq 2$, then every d-regular graph G satisfies $\chi'_{DP}(G) \geq d + 1$.

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Similarly, one can define the list b-fold coloring and the list fractional chromatic number $\chi^*_\ell(G)$ of a graph G.

A *b*-fold coloring of a graph assigns to each vertex a set of b colors, and color sets assigned to adjacent vertices are disjoint. The *b*th chromatic number $\chi_b(G)$ of a graph G is the minimum total number of colors needed in a b-fold coloring of G.

The fractional chromatic number $\chi^*(G)$ of a graph G is

$$\inf_{b \geq 1} \frac{\chi_b(G)}{b}.$$

KNOWN: (a) There is b such that $\chi^*(G) = \frac{\chi_b(G)}{b}$;
(b) $\chi^*(G)$ is rational.

Similarly, one can define the list b-fold coloring and the list fractional chromatic number $\chi^*_\ell(G)$ of a graph G.

Theorem 17 [Alon–Tuza-Voigt] $\chi^*_\ell = \chi^*(G)$ for every graph G.

A b-fold (L, H)-coloring of a graph G for each $v \in V(G)$ chooses b vertices in the list $L(v)$ of the cover graph H so that the chosen vertices are not adjacent to each other if they are in the lists of distinct vertices. They form a quasi-independent set in H.

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Naturally, the bth DP-chromatic number $\chi_{DP,b}(G)$ of a graph G is the minimum k such that G has a b-fold (L, H)-coloring for each choice of (L, H) with $|L(v)| \geq k$ for all $v \in V(G)$.

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The fractional DP-chromatic number $\chi^*_{DP}(G)$ of a graph G is

$$\inf_{b \geq 1} \frac{\chi_{DP,b}(G)}{b}.$$

Properties of $\chi^*_{DP}(G)$

Theorem 18 [Bernshteyn–A.K.–Zhu, 2017+] Let G be a connected graph. Then $\chi^*_{DP}(G) = 2$ if and only if G contains no odd cycles and has at most one even cycle. Furthermore, if G contains no odd cycles and exactly one even cycle, then $\chi^*_{DP}(G) = 2$, even though

$$\frac{\chi_{DP,b}(G)}{b} > 2 \quad \text{for all } b \in N.$$

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Theorem 19 [Bernshteyn–A.K.–Zhu, 2017+] If G is a graph of maximum average degree $d \geq 4$, then $\chi^*_{DP}(G) \geq d/(2 \ln d)$.

Properties of $\chi^*_{DP}(G)$

Theorem 20 [B.–K.–Z., 2017+] Suppose that a graph G has an acyclic orientation D such that
(a) $\Delta^+(D) \leq d$; and
(b) D has no odd cycles with exactly one edge in clockwise direction.
Then $\chi^*_{DP}(G) \leq (1 + o(1))d/\ln d$.

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Then $\chi^*_{DP}(G) \leq (1 + o(1))d/\ln d$.

Corollary 21 [B.-K.–Z., 2017+] If G is a d-degenerate bipartite graph, then $\chi^*_{DP}(G) \leq (1 + o(1))d/\ln d$.

Questions

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3. Do there exist graphs G with $\chi'_{DP}(G) \geq \Delta(G) + 2$?

4. We conjecture that for every multigraph G,

$$\chi_{DP}(\mathrm{Line}(G)) \leq 3\Delta(G)/2.$$