Finding and using expanders
in locally sparse graphs

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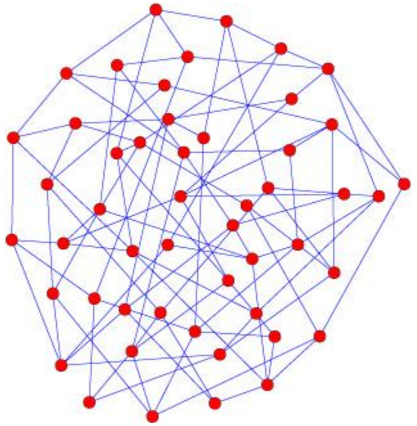
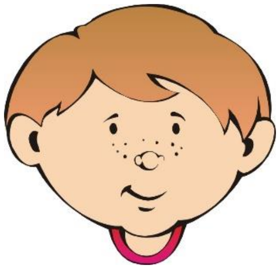


Synopsis 1

Expanders are a wonderful thing, theoretically and practically:

- lean but reliable networks;
- sparse graphs with small diameter;
- good for many extremal problems;
- etc.

Synopsis 2

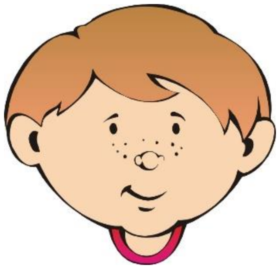


Nice, but how do I find one when I need one?

Well, there are:

- probabilistic constructions (Pinsker'73,...);
- explicit constructions (Margulis'73,...)

Synopsis 3

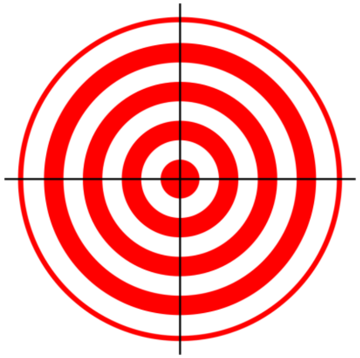


Sure, but are they omnipresent?
Which graphs contain expanders?

Ah, this is what you mean…
Then try locally sparse graphs

Synopsis 4

Main goals of this talk:



- to present a very simple sufficient condition
  for a large expanding subgraph in a graph;

- to exploit this conditions in applications
  (embedding minors, long cycles, positional games,…)

γ-expanders - definition

Notation: $G = (V, E)$ – graph, $W \subset V$ – vertex subset

$N_G(W) := \{v \in V \setminus W : v \text{ has a neighbor in } W\}$

– external neighborhood of W

Def.: $G = (V, E)$ – graph, $\gamma > 0$ – real

G is a γ-expander if

$$|N_G(W)| \geq \gamma |W|, \forall W \subset V, |W| \leq \frac{|V|}{2}$$

[mostly: $\Delta(G) = O(1)$, or $|E| = O(|V|)$]

Why bother?..

Nice properties of γ-expanders:

- connectedness;
- logarithmic diameter;
- rapid mixing of a random walk;
- all separators are large;
- embedding/minors;
- …

[more details later]

Some conditions are needed…

- Not every graph is an expander…

  (isolated vertices, multiple connected components, etc.)

- Not every graph contains a linearly sized expander…

- Not every graph contains a sizable expander…

  (e.g., planar graphs – hereditarily have sublinear separators)

Locally sparse graphs

Def.: $G = (V, E)$ – graph, $|V| = n$, $c_1 > c_2 > 1, \alpha > 0$ – reals
G is a (c_1, c_2, α)-graph if:

1. $|E| \geq c_1 n$;

2. $\forall U \subset V, |U| \leq \alpha n \Rightarrow e_G(U) < c_2|U|$.

[– small subsets are sizably sparser than the whole graph]

– Pretty mild/modest condition: G can have lots of isolated vertices, several large connected components,…

Where do locally sparse graphs come from?

- from random graphs naturally:

$G\left(n, \frac{c}{n}\right), c > 1$, is with high prob. (whp) locally sparse.

Prop: $c_1 > c_2 > 1$ – constants. Define: $\alpha = \left(\frac{c_2}{5c_1}\right)^{\frac{c_2}{c_2-1}}$.
$G \sim G\left(n, \frac{c_1}{n}\right)$ is whp s.t.
every $k \leq \alpha n$ vertices span $< c_2 k$ edges.

Proof: straightforward first moment/union bound:

$$\Pr[\exists \text{ a violating set}] \leq \sum_{k \leq \alpha n} \binom{n}{k} \binom{\binom{k}{2}}{c_2 k} p^{c_2 k} = \cdots = o(1)\,. \qquad \blacksquare$$

Can cap the degrees as well…

Prop: $G \sim G\left(n, \frac{c}{n}\right)$, $c > 1$ – const.

$\delta = \delta(c) > 0$ – small enough

whp every $\frac{\delta}{\log\frac{1}{\delta}} n$ vertices touch $\leq \delta n$ edges.

Proof: straightforward first moment/union bound:

$$\Pr[\exists \text{ a violating set}] \leq \binom{n}{\frac{\delta n}{\log\frac{1}{\delta}}} \binom{\frac{\delta n^2}{\log\frac{1}{\delta}}}{\delta n} p^{\delta n}$$

$$= \left[\left(\frac{e \log\frac{1}{\delta}}{\delta}\right)^{\frac{1}{\log\frac{1}{\delta}}} \cdot \frac{ec}{\log\frac{1}{\delta}}\right]^{\delta n} = o(1). \quad \blacksquare$$

Main result

Th.: $G = (V, E)$ – graph, $|V| = n$;

$\Delta > 0$ - integer; $c_1 > c_2 > 1, \alpha > 0$ – reals

Assume:

1. $|E| \geq c_1 n$;
2. $\forall U \subset V, |U| \leq \alpha n \Rightarrow e_G(U) < c_2 |U|$;
3. $\Delta(G) \leq \Delta$.

} $-(c_1, c_2, \alpha)$-graph

Then: G contains an induced subgraph $G^* = (V^*, E^*)$:

– $|V^*| \geq \alpha n$;

– G^* is a γ-expander, for $\gamma = \frac{c_1 - c_2}{\Delta \left\lceil \log_2 \frac{1}{\alpha} \right\rceil}$.

Remarks:

1. Get a linearly sized induced expander;
2. Condition $\Delta(G) = O(1)$ – to trade edge expansion for vertex expansion.

Proof sketch 1

H := minimal by inclusion non-empty induced subgraph of G

s.t. $\frac{|E(H)|}{|V(H)|} \geq c_1$

[exists by Prop. 1 of G]

Properties of $H = (U, F)$:

1. $|U| \geq \alpha n$ [by Prop. 2 of G];

2. $\forall W \subseteq U, W$ is incident to $\geq c_1|W|$ edges of H

[otherwise delete W, get a smaller H' of density $\geq c_1 - \perp$]

[3. H is connected]

Proof sketch 2

$\delta > 0 :=$ small constant ($\delta \ll c_1 - c_2$)

a) $\forall W \subseteq U, |W| \leq \alpha n$

$$\Rightarrow e_H(W) < c_2|W|$$

$$e_H(W) + e_H(W, \bar{W}) \geq c_1|W|$$

$$\overline{}$$

$e_H(W, \bar{W}) \geq (c_1 - c_2)|W|$ – edge boundary is substantial

b) If: $\forall W \subset U, \alpha n \leq |W| \leq \frac{|U|}{2}, e_H(W) \leq (c_1 - \delta)|W|$

$$\Rightarrow e_H(W, \bar{W}) \geq c_1|W| - (c_1 - \delta)|W| = \delta|W|$$

Proof sketch 3

Otherwise: $\exists W \subset U, \alpha n \leq |W| \leq \frac{|U|}{2}, e_H(W) \geq (c_1 - \delta)|W|$

- give up a bit on global density (only δ);
- cut the order by the factor ≥ 2.

Consider now $G[W]$, repeat…

Stops after $O(\log \frac{1}{\alpha})$ steps. ■

Algorithmic side 1

(Eigenvalues and expansion:

Notation:

$G = (V, E)$ – graph, $V = [n]$

$W \subseteq V$, $vol_G(W) := \sum_{v \in V} \deg_G(v)$ – volume of W

$A = (a_{ij})$ – adjacency matrix of G;

$D := diag\ (d(1), \ldots, d(n))$ – degree matrix of G;

$\mathcal{L} := I_n - D^{-1/2} A D^{-1/2}$ – normalized Laplacian of G;

$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ - eigenvalues of $\mathcal{L}$;

$\lambda := \lambda_1$ – eigenvalue of G

(can be computed in time polynomial in n)

Algorithmic side 2

Cheeger:

Notation:

$$h(G) := \min_{\emptyset \neq W \subset V} \frac{e_G(W, V \setminus W)}{\min(vol_G(W), vol_G(V \setminus W))}$$

– Cheeger constant of G

Cheeger inequality:

$$\frac{h^2(G)}{2} \leq \lambda(G) \leq 2h(G)$$

Constructive: can find in time $poly(n)$ a subset $W \subset V, vol_G(W) \leq \frac{1}{2} vol_G(V)$, satisfying $e_G(W, V \setminus W) \leq \sqrt{2\lambda(G)} \cdot vol_G(W)$.

Algorithmic side 3

Iterate:

G_i ≔ current graph

Compute $\lambda(G_i)$

$\lambda(G_i)$ – large? $\Rightarrow$ G_i is a good expander by Cheeger

$\lambda(G_i)$ – small? $\Rightarrow$ find a non-expanding set W_i,

proceed as in the existential proof

Th.: $\Delta > 0$ - integer; $c_1 > c_2 > 1, \alpha > 0$ – reals

$\exists\, \gamma = \gamma(c_1, c_2, \Delta) > 0$ and a poly. time algorithm s.t.

given $G = (V, E)$, $|V| = n$, $\Delta(G) \leq \Delta, \frac{|E|}{|V|} \geq c_1$,

finds:

- a subset $W \subset V, |W| \leq \alpha n, e_G(W) \geq c_2|W|$ (small dense set)

or

- induced γ-expander $G^* \subseteq G$ on $\geq \alpha n$ vertices (large expander).

Applications 1: linearly sized expander in the supercritical random graph

Cor. 1: $G \sim G\left(n, \frac{1+\epsilon}{n}\right), \epsilon > 0$ – constant,

whp contains a linearly sized γ-expander.

Proof: Look at the giant component C_1 of G:

whp: 1. $|C_1| = \Theta(n)$;

2. $\frac{e(C_1)}{|C_1|} \geq 1 + \delta,\ \delta = \delta(\epsilon) > 0$ (positive rel. excess);

3. $G[C_1]$ – locally sparse;

4. can handle few high degree vertices.

$\Rightarrow G[C_1]$ contains a linearly sized expander (⇐main th.). ■

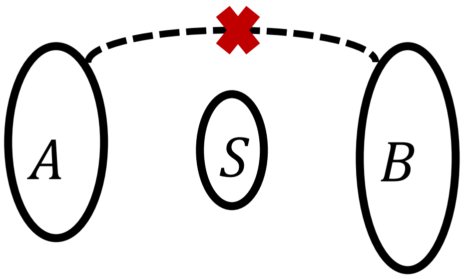
Separators

Def.: $G = (V, E)$ – graph, $|V| = n$

$S \subset V$ is a separator in G if

there is a partition $V = A \cup S \cup B$

s.t. $|A|, |B| \leq \frac{2n}{3}$, no edges between A and B



Meaning: all separators large

$\Rightarrow$ costly to break G into large not connected pieces

Prop.: $G = (V, E)$ – γ-expander on n vertices, S – separator in G

$\Rightarrow |S| \geq \frac{\gamma n}{3(1+\gamma)}$.

Minors

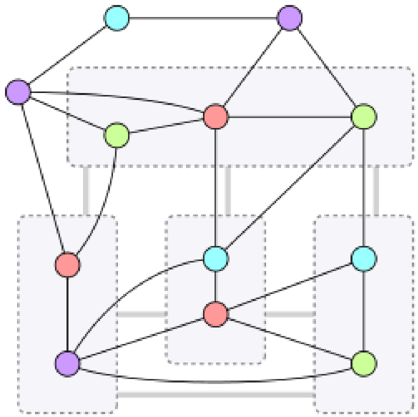
Def.: $G = (V, E)$ – graph

G contains a minor of $H = (\{v_1, \ldots, v_t\}, F)$ if

$\exists$ disjoint $V_1, \ldots, V_t \subset V(G)$ s.t.:

- $G[V_i]$ connected;
- $(v_i, v_j) \in F \Rightarrow G$ has an edge between V_i, V_j.

Ex.:



- minor of K_4

Applications 2: embedding minors

Kawarabayshi, Reed'10: large complete minor or small separator:

$G = (V, E), |V| = n; h = h(n)$

G has a minor of K_h or a separator $S, |S| = O(h\sqrt{n})$

Cor. 2: $\Delta > 0$ - integer; $c_1 > c_2 > 1, \alpha > 0$ – reals
$\exists\ c = c(c_1, c_2, \Delta, \alpha) > 0$ s.t.
$\forall\ (c_1, c_2, \alpha)$ –graph $G, |V(G)| = n, \Delta(G) \leq \Delta$
$\Rightarrow$ G contains a minor of $K_{c\sqrt{n}}$.

Cor. 3: $G \sim G\left(n, \frac{1+\epsilon}{n}\right), \epsilon > 0$ – constant,
whp contains a minor of $K_{c\sqrt{n}}$.

– recovers the result of Fountoulakis, Kühn, Osthus'08.

Applications 3: long cycles

Th. [K'18+]: $G - \gamma$-expander on n vertices

$\Rightarrow G$ contains a cycle of length $\geq \frac{\gamma n}{4}$.

Cor. 4 : $G -$ (c_1, c_2, α)-graph, $|V(G)| = n, \Delta(G) \leq \Delta$

$\Rightarrow G$ contains a linearly long cycle.

– very convenient tool for finding long cycles.

Applications 4: positional games

Biased Maker-Breaker games on $E(K_n)$

Setting: board = $E(K_n)$

two players: Maker, Breaker, alternately claim unoccupied edges

bias $b \geq 1$: Maker claims 1 edge each turn,
Breaker claims b edges

Maker wins: iff in the end his graph has target property $\wp$

Making cycles in Maker-Breaker games

Th: $\forall \epsilon > 0\ \exists n_0\ \forall n \geq n_0$

$b \leq (1-\epsilon)\frac{n}{2} \Rightarrow$ Maker can create a linearly long cycle in the $(1{:}\,b)$ MB game on $E(K_n)$

Proof idea: In the end $|M| = \frac{\binom{n}{2}}{b+1} \approx (1+\epsilon)n$

Maker plays randomly in the first $\left(1+\frac{\epsilon}{2}\right)n$ rounds, gets M_0;

each edge of M_0 is claimed by Maker with prob. $\Theta(n^{-2})$, indep. of previously claimed edges

Making cycles in Maker-Breaker games (cont.)

Proof idea (cont.):

With positive probability (actually whp):

- $|E(M_0)| = \left(1 + \frac{\epsilon}{2}\right) n$;

- for some $\alpha = \alpha(\epsilon) > 0, \forall U \subset [n], |U| \leq \alpha n$

  $\Rightarrow U$ spans $\leq \left(1 + \frac{\epsilon}{4}\right) |U|$ edges of M_0 (=locally sparse);

- rel. few vertices of high degree in M_0 (can get $\Delta(M_0') = O(1)$)

$\Rightarrow$ apply Cor. 4 to get a linearly long cycle. ■

Making cycles in Maker-Breaker games (cont.)

Remarks:

1. by Bednarska-Pikhurko'05:
   $b \geq \left\lfloor \frac{n}{2} \right\rfloor - 1 \Rightarrow$ Breaker can force Maker's graph to be acyclic
   $\Rightarrow$ our result is asympt. optimal;

2. for long paths, K., Sudakov'13:
   $b \leq (1-\epsilon)n$
   $\Rightarrow$ Maker can create a linearly long path
   - substantial difference between paths, cycles here.

The end

Конец