Searching stochastic equilibria in transport networks by universal primal-dual gradient method

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Outline

1. Problem Statement. Solution.
2. Computational complexity
3. Results of the experiments

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$$\text{time costs: } \tau_e(f_e),\ G_p(f) = \sum_{e \in E} \tau_e(f_e)\delta_{ep}$$

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The primal problem

$$\sum_{e\in E}\sigma_e(f_e)+\gamma\sum_{w\in OD}\sum_{p\in P_w}x_p\ln(\frac{x_p}{d_w})\longrightarrow\min_{f=\Theta x, x\in X}$$

$$\text{где } \gamma>0,\ \sigma_e(f_e)=\int_0^{f_e}\tau_e(z)dz$$

$$\Theta=\|\delta_{ep}\|_{e\in E, p\in P},\ \delta_{ep}=Ind\{e\in p\}$$

$$X=\{x\geq 0:\ \sum_{p\in P_w}x_p=d_w,\ w\in OD\}$$

$$\text{BPR-function: } \tau_e(f_e)=\bar{t}_e\left(1+\rho\left(\frac{f_e}{\bar{f}_e}\right)\right)^{\frac{1}{\mu}}$$

The construction of the dual problem

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$$= \max_{t\in dom\sigma^*} -\left[\sum_{e\in E}\sigma_e^*(t_e) + \gamma\psi\left(t/\gamma\right)\right] = \max_{t\in dom\sigma^*} -Q(t)$$

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$$\text{duality gap}: \; d^{(N)} = F(f^{(N)}, x^{(N)}) + Q(t^{(N)})$$

The dual problem

$$Q(t) = \gamma\psi\left(t/\gamma\right) + \sum_{e\in E}\sigma_e^*(t_e) \longrightarrow \min_{t\in dom\sigma^*}$$

$$\text{где } \psi(t) = \sum_{w\in OD} d_w\psi_w(t),\ \psi_w(t) = \ln\left(\sum_{p\in P_w}\exp\left(-\sum_{e\in E}\delta_{ep}t_e\right)\right)$$

$$f = -\nabla\gamma\psi(t/\gamma),\ x_p = d_w\frac{\exp\left(-\frac{1}{\gamma}\sum_{e\in E}\delta_{ep}t_e\right)}{\sum_{q\in P_w}\exp\left(-\frac{1}{\gamma}\sum_{e\in E}\delta_{eq}t_e\right)},\ p\in P_w$$

$$\sigma_e^*(t_e) = \bar{f}_e\left(\frac{t_e-\bar{t}_e}{\bar{t}_e\rho}\right)^{\mu}\frac{(t_e-\bar{t}_e)}{1+\mu}$$

Numerical solution of the problem

$$Q(t) = \underbrace{\Phi(t)}_{\gamma\psi(t/\gamma)} + \underbrace{h(t)}_{\sum_{e\in E}\sigma_e^*(t_e)} \longrightarrow \min_{t\in\sigma^*}$$

Let

$$\phi_0(t) = \alpha_0 \left[\Phi(y^0) + \left\langle \nabla\Phi(y^0), t - y^0 \right\rangle + h(t) + \frac{1}{2}\left\| t - y^0 \right\|_2^2\right]$$

$$\phi_{k+1}(t) = \phi_k(t) + \alpha_{k+1}\left[\Phi(y^{k+1}) + \left\langle \nabla\Phi(y^{k+1}), t - y^{k+1} \right\rangle + h(t)\right]$$

Numerical solution of the problem

Adaptive method of similar triangles (AMST):

1. $L^0_{k+1} = L^{j_k}_k/2, \ j_{k+1} = 0$

2. $$\begin{cases} \alpha_{k+1} := \frac{1}{2L^{j_{k+1}}_{k+1}} + \sqrt{\frac{1}{4\left(L^{j_{k+1}}_{k+1}\right)^2} + \frac{A_k}{L^{j_{k+1}}_{k+1}}}, \ A_{k+1} := A_k + \alpha_{k+1} \\ y^{k+1} := \frac{\alpha_{k+1}u^k + A_k t^k}{A_{k+1}}, \ u^{k+1} := argmin_{t \in dom \sigma^*} \phi_{k+1}(t) \\ t^{k+1} := \frac{\alpha_{k+1}u^{k+1} + A_k t^k}{A_{k+1}} \end{cases}$$

While

$\Phi(y^{k+1}) + \left\langle \nabla\Phi(y^{k+1}), t^{k+1} - y^{k+1} \right\rangle + \frac{L^{j_{k+1}}_{k+1}}{2} \left\| t^{k+1} - y^{k+1} \right\|_2^2 + \frac{\alpha_{k+1}}{2A_{k+1}}\varepsilon < \Phi(t^{k+1})$

do $j_{k+1} := j_{k+1} + 1, \ L^{j_{k+1}}_{k+1} = 2^{j_{k+1}} L^0_{k+1}$

3. $k := k + 1$ — next iteration.

P_{ij}^{l} - set of paths from i to j of l edges
$\tilde{P}_{ij}^{l}$ - set of paths from i to j of not more than l edges
$\tilde{P}_{ij}^{l+1} = \tilde{P}_{ij}^{l} \cup P_{ij}^{l+1}$
$l \leq H,\ H = O(\sqrt{n})$, n - number of edges of the graph

Calculating the value of the function $\psi(t/\gamma)$

$$\begin{cases} a_{ij}^{l}(t) = \gamma\psi_{P_{ij}^{l}}(t/\gamma) = \gamma \ln\left(\sum_{p\in P_{ij}^{l}} \exp\left(-\sum_{e\in E}\delta_{ep}t_e/\gamma\right)\right) \\ b_{ij}^{l}(t) = \gamma\psi_{\tilde{P}_{ij}^{l}}(t/\gamma) = \gamma \ln\left(\sum_{p\in \tilde{P}_{ij}^{l}} \exp\left(-\sum_{e\in E}\delta_{ep}t_e/\gamma\right)\right) \end{cases}$$

$$a_{ij}^{1}(t) = b_{ij}^{1}(t) = \begin{cases} -t_e, e = (i\to j)\in E \\ -\infty, e = (i\to j)\notin E \end{cases}$$

$j\in V, l = \overline{1, H-1}$:

$$\begin{cases} a_{ij}^{l+1}(t) = \gamma \ln\left(\sum_{k:e=(k\to j)\in E} \exp\left(\left(a_{ik}^{l}(t) - t_e\right)/\gamma\right)\right) \\ b_{ij}^{l+1}(t) = \gamma \ln\left(\exp\left(b_{ij}^{l}(t)/\gamma\right) + \exp\left(a_{ij}^{l+1}(t)\right)\right) \end{cases}$$

Calculating the value of the function $\psi(t)$

Memory: $O(SHn)$, S - number of origins.
Time: $O(SHn)$.

Calculating the gradient $\nabla\psi(t/\gamma)$

$$\gamma\psi^i(t/\gamma) = \sum_{j:w=(i,j)\in OD} d_w \gamma\psi_w(t/\gamma), \quad \frac{\partial\psi^i}{\partial b_{ij}^H} = d_{ij}, \ \frac{\partial\psi^i}{\partial a_{ij}^H} = 0$$

$$\begin{cases} \frac{\partial\psi^i}{\partial b_{ij}^l} = \frac{\partial\psi^i}{\partial b_{ij}^{l+1}} \frac{\partial b_{ij}^{l+1}}{\partial b_{ij}^l}, \ j \in V, \ l = \overline{H-1,1} \\ \frac{\partial\psi^i}{\partial a_{ij}^l} = \frac{\partial\psi^i}{\partial b_{ij}^l} \frac{\partial b_{ij}^l}{\partial a_{ij}^l} + \sum_{k:e=(j\to k)\in E} \frac{\partial\psi^i}{\partial a_{ik}^{l+1}} \frac{\partial a_{ik}^{l+1}}{\partial a_{ij}^l} \end{cases}$$

$$\frac{\partial\psi^i}{\partial t_e} = \sum_{l=0}^{H-1} \frac{\partial\psi^i}{\partial a_{ij}^{l+1}} \frac{\partial a_{ij}^l}{\partial t_{e=(k,j)}}$$

Calculating the $\nabla\psi(t)$

Memory: $O(SHn)$, S — number of origins.
Time: $O(SHn)$.

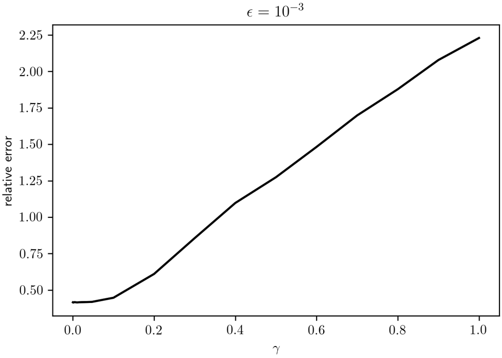
Theorem

AMST with stop criteria $d(t^{(N)} \leq \varepsilon)$ (d — duality gap) is guaranteed to stop, reaching ε–accuracy $(F(t^{(N)}) - F(t_*) \leq \varepsilon)$, when the number of arithmetic operations made is no more than the number shown in the table:

Run-time	$\gamma > 0$	$\gamma \longrightarrow +0$
$\mu \longrightarrow +0,\ \mu > 0$	$O(SHn\sqrt{\frac{Hd\bar{R}^2}{\gamma\varepsilon}})$	$O(Sn\ln n\frac{Hd^2\bar{R}^2}{\varepsilon^2})$

Anaheim Network

Edges: 914
Vertices: 416
Trips: 104,694.40



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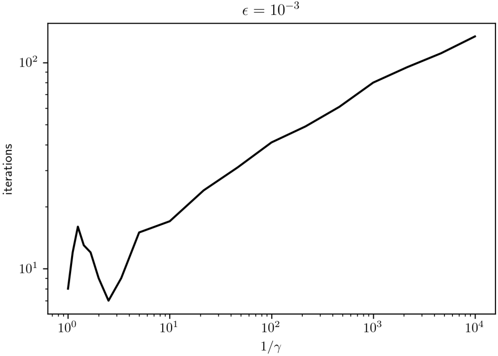

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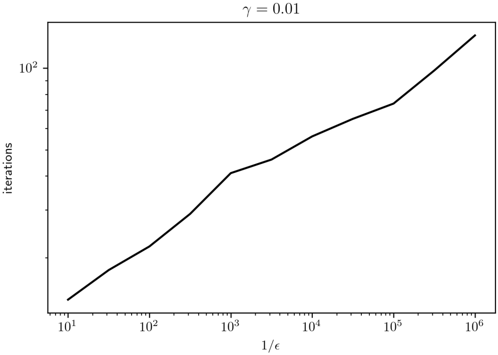

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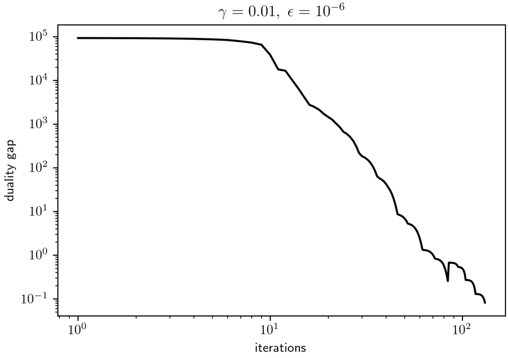

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Universalnyi metod poiska ravnovesii i stohasticheskih ravnovesii v transportnyh setyah[Universal similar triangulars method for searching equilibria in traffic flow distribution models]. — ZhVM MF[Comp. Math. Math. Phys.], 2019 — Vol. 58. (in Russian)

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