BOUNDARY AND MINIMAL HARD CLASSES FOR ALGORITHMIC GRAPH PROBLEMS

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Hereditary graph classes

Definition

A simple graph is a non-oriented finite graph without loops and multiple edges.

Definitions

A class is any set of simple graphs, closed under isomorphism. A class of graphs is hereditary if it is closed under deletion of vertices.

Properties and notation

For any hereditary class $\mathcal{X}$, there is a unique set $\mathcal{S}$ of minimal (under deletion of vertices) graphs, not belonging to $\mathcal{X}$; written as $\mathcal{X} = Free(\mathcal{S})$. Graphs in $\mathcal{S}$ are called forbidden induced subgraphs for $\mathcal{X}$.

Examples

- $\mathcal{F}orests = Free(\{C_3, C_4, C_5, \ldots\})$
- $\mathcal{B}ipartite = Free(\{C_3, C_5, C_7, \ldots\})$

Definition

A hereditary class is called finitely defined if the set of its forbidden induced subgraphs is finite.

Examples

- The classes $\mathcal{F}orests, \mathcal{B}ipartite$ are not finitely defined.
- For any fixed d, the class $\mathcal{D}eg(d)$ of all graphs with maximum degree of vertices at most d is finitely defined.
- The class of all line graphs is finitely defined.

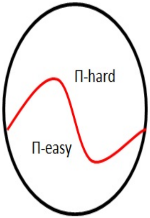
Main problem

HOW TO CLASSIFY HEREDITARY GRAPH CLASSES WITH RESPECT TO THE COMPLEXITY OF A GIVEN GRAPH PROBLEM?

Π-easy and Π-hard graph classes

Definitions

Let Π be a NP-complete graph problem. A hereditary class, for which Π is polynomial-time solvable, is called Π-easy. A hereditary class, for which Π is NP-complete, is called Π-hard.



Maximal easy and minimal hard graph classes

Result [a proof is trivial]

For any NP-complete graph problem Π, there is not a maximal Π-easy class.

Comment

Minimal hard classes exist for some problems and do not exist for another.

Result [a proof is trivial]

The set of complete graphs is a minimal hard class for the travelling salesman problem.

Result [D.S.M.'09]

For any k, there are not minimal hard classes for the vertex and for the edge k-colourability problems.

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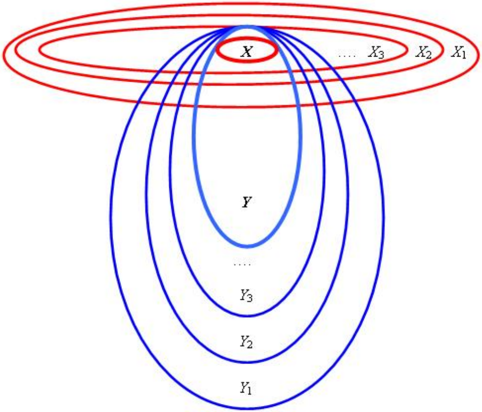
The notion of a boundary graph class and its significance



Definition

A class of graphs $\mathcal{X}$ is called Π-limit if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \mathcal{X}_3 \supseteq \ldots$ of Π-hard graph classes, such that $\mathcal{X} = \bigcap_{i=1}^{\infty} \mathcal{X}_i$.

Definition

A minimal Π-limit class is called Π-boundary.

Result [V.E. Alekseev'04]

A finitely defined class is Π-hard if and only if it contains a Π-boundary class.

Comment 1

Knowledge of all Π-boundary classes admits to completely describe all finitely defined Π-hard classes.

Comment 2

Minimal hard classes are elements of the border between polynomial-time solvability and NP-completeness in the family of hereditary classes.

Boundary classes are a tool for classification of finitely defined classes.

Comment 3

For any NP-complete graph problem, there is a boundary class.

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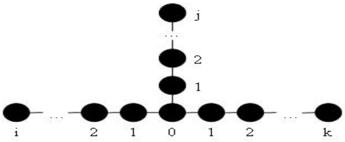
Known boundary classes for some graph problems

Definition

The class $\mathcal{T}$ consist of all graphs, whose all connected components are trees with at most three leaves.

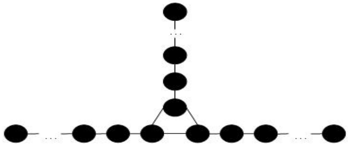


Result [V.E. Alekseev'04]

If $P \neq NP$, then the class $\mathcal{T}$ is boundary for the independent set problem.

Definition

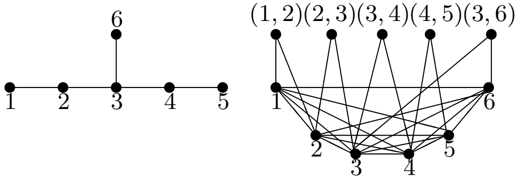
The class $\mathcal{D}$ consists of line graphs to graphs in $\mathcal{T}$.



Definition

For a given graph $G = (V, E)$, a graph $Q(G)$ has:

- the set of vertices $V \cup E$
- the set of edges $\{(v_i, v_j) : v_i, v_j \in V\} \cup \{(v, e) : v \in V, e \in E, v \text{ is incident to } e\}$.



Definition

The hereditary closure of $\{Q(G) : G \in \mathcal{T}\}$ is denoted by $\mathcal{Q}$.

Result [V.E. Alekseev, D.V. Korobitsyn, V.V. Lozin'04]

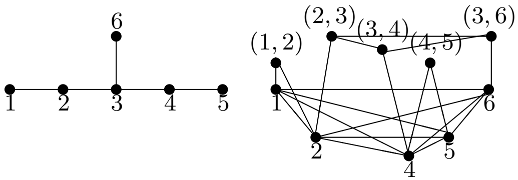
If $P \neq NP$, then the classes $\mathcal{T}, \mathcal{D}, \mathcal{Q}$ are boundary for the dominating set problem.

Definition

Let $G = (V, E)$ be a subcubic graph without adjacent degree three vertices, V' be the set of its degree three vertices and $V'' \triangleq V(G) \setminus V'$. A vertex $x \in V'$ is incident to edges $e_1(x), e_2(x), e_3(x)$ in the graph G.

A graph $\mathcal{Q}^*(G)$ has:

- the set of vertices $V'' \cup E$
- the set of edges $\{(v_i, v_j) : \ v_i, v_j \in V''\} \cup \{(v, e) : \ v \in V'', e \in E, v$ is incident to $e\} \cup \bigcup_{x \in V'} \{(e_1(x), e_2(x)), (e_1(x), e_3(x)), (e_2(x), e_3(x))\}$.



Definition

The hereditary closure of $\{Q^*(G) : G \in \mathcal{T}\}$ is denoted by $\mathcal{Q}^*$.

Result [D.S.M.'16]

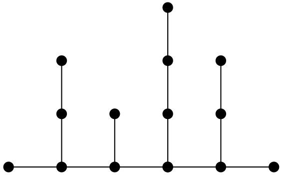
If $P \neq NP$, then the class $\mathcal{Q}^*$ is boundary for the dominating set problem.

Definition

A caterpillar is a subcubic graph, obtained by coinciding ends of simple paths with vertices of a simple path.

The class $\mathfrak{S}_1$ is the hereditary closure of the set of all caterpillars, the class $\mathfrak{S}_2$ is the hereditary closure of the set of all graphs, obtained by «inscribing» a triangle into every degree three vertex in all caterpillars.

The class $co(\mathcal{D})$ is the set of complement graphs to graphs in $\mathcal{D}$.



Result [V.V. Lozin, N. Korpelainen, D.S.M., A. Tiskin'11]

If $P \neq NP$, then the classes $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are boundary for the Hamiltonian cycle problem.
If $P \neq NP$, then the class $co(\mathcal{D})$ is boundary for the chromatic number problem.

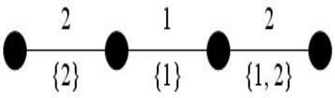
The first example of complete descriptions of boundary systems

Definition

Let a graph G with a set E of edges and a set $\mathfrak{L} = \{\mathcal{L}(e) : \ e \in E\}$, where each $\mathcal{L}(e)$ is a finite set of naturals, be given. An edge coloring of the graph G is called $\mathfrak{L}$-ranking of G if the following properties are true:

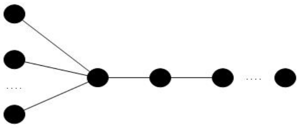
- $c(e) \in \mathcal{L}(e)$, for any edge e;
- if $c(e_1) = c(e_2)$, $e_1 \neq e_2$, then every path, connecting e_1 and e_2, contains an edge e, such that $c(e) > c(e_1)$.

The edge list-ranking problem is to determine, for given G and $\mathfrak{L}$, whether there is a $\mathfrak{L}$-ranking of G or not.



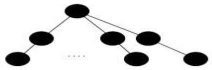
Definition

The class *Comet* is the hereditary closure of the set of graphs of the form:



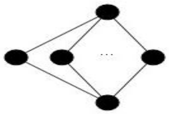
Definition

The class *Star* is the hereditary closure of the set of graphs of the form:



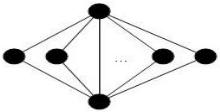
Definition

The class *Bat* is the hereditary closure of the set of graphs of the form:



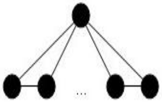
Definition

The class *Comb* is the hereditary closure of the set of graphs of the form:



Definition

The class *Camomile* is the hereditary closure of the set of graphs of the form:

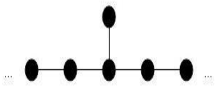


Definition

The class *Cliques* is the set of complete graphs.

Definition

The class $\tilde{\mathcal{T}}$ is the hereditary closure of the set of graphs, whose all connected components have the form:



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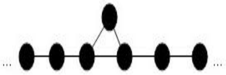
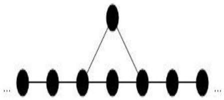
The class $\tilde{\mathcal{D}}$ is the hereditary closure of the set of graphs, whose all connected components have the form:



Definition

The class $\hat{\mathcal{T}}$ is the hereditary closure of the set of graphs, whose all connected components have the form:



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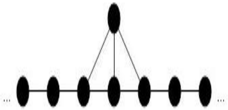

Result [D.S.M.'13]

The boundary system for the edge list-ranking problem is constituted by the classes $Comet, Star, \mathcal{B}at, Comb, Camomile, Cliques, \tilde{\mathcal{T}}, \tilde{\mathcal{D}}, \hat{\mathcal{T}}, \hat{\mathcal{D}}$.

On difficulties for obtaining complete descriptions of boundary systems for some graph problems

Comment

A boundary system can be too complex that is impossible to obtain a complete its description.

Result [D.S.M.'09 and '12]

For any $k \geq 3$, the sets of boundary classes for the vertex and edge k-colourability problems have the continuum cardinality.

Some references

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THANK YOU FOR YOUR ATTENTION!