Two results on trees with extremal number of independent sets

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- An *independent set* in a graph is a subset of its pairwise nonadjacent vertices.
- A *maximal* (respectively, *maximum*) independent set is an independent set, which is maximal by inclusion (respectively, by the cardinality).
- The number of all (respectively, maximal or maximum) independent sets in a graph G is denoted by $\mathrm{i}(G)$ (respectively, by $\mathrm{mi}(G)$ or $\mathrm{xi}(G)$) .
- We will use the abbreviation «i.s.» for the term «independent set».

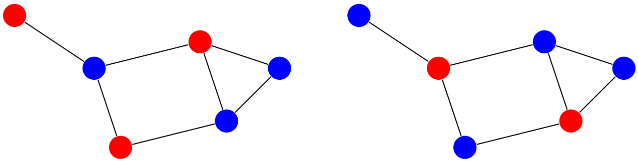

Fig. 1. Maximum i.s. and maximal i.s. which is not maximum.

Introduction

There are many papers on counting all, maximal or maximum i.s. in various classes of graphs.

- In 1965, J. W. Moon and L. Moser have found the maximum number of maximal i.s. in the class of all n-vertex graphs.
- Depending on the residue modulo 3, the extremal graphs are isomorphic to
  - mK_3, if $n = 3m$
  - $2K_2 \cup (m-1)K_3, K_4 \cup (m-1)K_3$, if $n = 3m+1$
  - $K_2 \cup mK_3$, if $n = 3m+2$
- Obviously, every maximal i.s. in those extremal graphs is also a maximum independent set.
- Hence, the problems of finding the maximum numbers of maximal and maximum i.s. in the class of all n-vertex graphs are the same.

- In 1991, J. Zito has found the maximal number of maximum i.s. in the class of all n-vertex trees.
- Later, in 2000, M. J. Jou and C. J. Chang have proposed a more elegant solution for this problem.
- Depending on the residue modulo 2, the extremal trees look like

Fig. 2. Extremal n-vertex trees.

Related results

- In 2007, C. Heuberger and S. Wagner have described, for all n and d, all n-vertex trees of maximum vertex degree d, which have the maximal number of all i.s.
- However, the maximal number of i.s. for this tree subclass was not determined as an explicit function on n and d.
- For all n and d, the extremal tree is unique.

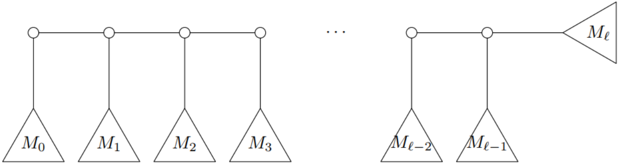

Fig. 3. The structure of the extremal tree for the case $d = 3$.
By M_i the complete binary trees of the height $i - 1$ or $i + 1$ are denoted.

Related results

- In 2011, the following generalization of the previous result was obtained.
- E. Andriantiana has proposed a way to construct a tree with the given degree sequence, which has the maximal number of all i.s.

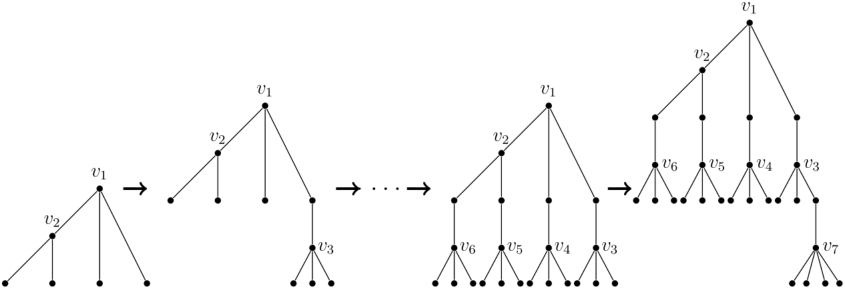

Fig. 4. Constructing the extremal tree with the degree sequence $(5, 4, 4, 4, 4, 3, 3, 2, 2, 2, 2, 2)$

- A tree is called (i, d, n)-maximal (respectively (xi, d, n)-maximal) if it has the maximal number of i.s. (respectively, maximum i.s.) among all n-vertex trees of maximum vertex degree at most d.

For any n and d, we describe all (xi, d, n)-maximal trees.

- For any $d \geq 3$ and $n = 2k$, the extremal tree is unique.
- For some $d > 3$ and $n = 2k + 1$, uniqueness does not hold.
- For $d = 3$ and any odd $n > 5$, there are exactly $\lceil \frac{n-3}{4} + 1 \rceil$ different extremal trees.

The extension of a graph

- Denote by $\mathrm{ext}(G)$ the graph, obtained by adding a leaf to every vertex of a graph G.
- We call $\mathrm{ext}(G)$ *the extension* of a graph G.

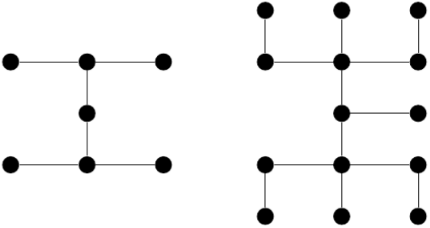

Fig. 5. The tree and its extension.

A property of the graph extension operation

The extension of a graph has the following important property:

Lemma 1.

For any graph G and its extension $ext(G)$, the equality $i(G) = xi(ext(G))$ holds.

- We show that every $(xi, d, 2k)$-maximal tree is the extension of the $(i, d-1, k)$-maximal tree.
- Thus, it is sufficient to describe all (i, d, n)-maximal trees, which has been done by Heuberger and Wagner in 2007.

A structure of the $(xi, d, 2k)$-maximal trees

Denote by $T_{d,n}$ the n-vertex tree of degree no more than d, which has the maximal number of all i.s. among all such trees.

Theorem 2.

There exists the unique $(xi, d, 2k)$-maximal tree. It is isomorphic to the tree $ext(T_{d-1,k})$.

Example. The figure below shows the $(xi, 4, 12)$-maximal tree, obtained from the $(i, 3, 6)$-maximal tree.

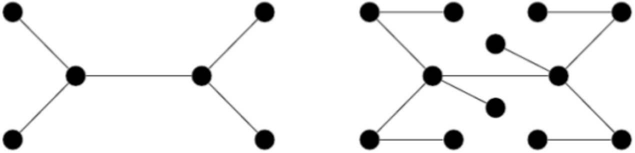

Fig. 6. The trees $T_{3,6}$ and $ext(T_{3,6})$.

The d-upbuilding operation

- Denote by S_p the graph obtained by subdividing $p-2$ edges of the graph $K_{1,p}$.
- We call the *d-upbuilding* of a tree T a tree, obtained by connecting the central vertex of the graph S_{d-1} with some vertex of the tree $ext(T)$, which has degree no more than $d-1$.

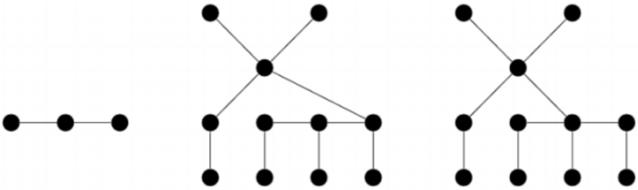

Fig. 7. The tree P_3 and two its different 4-upbuildings.

A structure of $(xi, d, 2k+1)$-maximal trees

Lemma 3.

The forest $(s-1)K_1 \cup T_{d,n-s+1}$ has the maximal number of all i.s. among all n-vertex forests with at most s connected components and the maximal degree at most d.

Theorem 4.

If $k \geq d-1$, then the set of all $(xi, d, 2k+1)$-maximal trees is equal to the set of all possible d-upbuildings of the tree $T_{d-1,k-d+2}$.
If $1 \leq k < d-1$, then there is the unique $(xi, d, 2k+1)$-maximal, which is isomorphic to S_{k+1}.

- Thus, a $(xi, d, 2k+1)$-maximal tree may not be unique.
- It is difficult to calculate the number of all $(xi, d, 2k+1)$-maximal trees for any $d > 3$.
- However, the case $d = 3$ is simple.

Case $d = 3$

- The structure of $(xi, 3, n)$-extremal trees is simple and it is possible to completely describe them.
- We use the notation $R_k = ext(P_k)$

Lemma 5.

The following statements hold:
1. There exists the unique $(xi, 3, 2k)$-maximal tree, which is isomorphic to the graph R_k.
2. For any $k \geq 3$, there exist exactly $\lceil \frac{k+1}{2} \rceil$ non-isomorphic $(xi, 3, 2k+1)$-maximal trees.

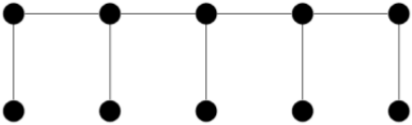

Fig. 8. The graph R_5.

Twin-leaves and their properties. The second result.

- Two leaves in a graph are called *twin-leaves* if they have a common neighbour.
- Obviously, if we remove one of the twin-leaves in a graph G, we obtain a graph G' such that $mi(G) = mi(G')$
- The lower bound on the number of maximal i.s. in the class of n-vertex trees is trivial, since the n-vertex star has 2 maximal i.s. for any $n \geq 2$.

We obtain the exact lower bound to the number of maximal i.s. in the class of n-vertex trees without twin-leaves and describe all the corresponding extremal trees.

Sets $u(T_1, \ldots, T_n)$

- A vertex in a tree is called *penultimate* if it is adjacent to a leaf.
- Denote by $u(T_1, T_2)$ the graph obtained from the disjoint union $T_1 \cup T_2$ by adding a new vertex of the degree 2, which is adjacent to some penultimate vertex in T_1 and to some penultimate vertex in T_2.
- It is easy to check that for any $T \in u(T_1, T_2)$ the equality $mi(T) = mi(T_1) \cdot mi(T_2)$ holds.
- Furthermore, we recursively define a set $u(T_1, T_2, \ldots, T_{n-1}, T_n)$ which consists of all trees T such that $T \in u(T', T_n)$, where $T' \in u(T_1, T_2, \ldots, T_{n-1})$.
- Recall that R_n is a graph obtained by adding a leaf to every vertex of the path P_n.

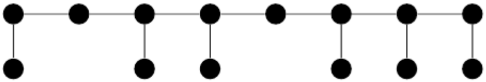

Fig. 9. A tree from the set $u(R_1, R_2, R_3)$

Classes $\mathcal{R}(a, b, c, d)$

- Now we define the classes which contain all the trees with the minimal number of maximal i.s.
- A tree belongs to the class $\mathcal{R}(a, b, c, d)$ if and only if it belongs to some set $u(T_1, \ldots, T_m)$ and exactly a, b, c and d trees T_i are isomorphic to the graphs R_1, R_2, R_3 and R_4, respectively.

Example. The following tree belongs to the class $\mathcal{R}(1, 1, 1, 0)$. Every tree from this class has $mi(R_1) \cdot mi(R_2) \cdot mi(R_3) = 2 \cdot 3 \cdot 5 = 30$ maximal i.s.

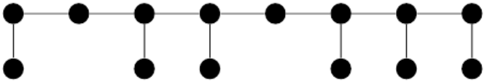

Fig. 10. A tree from the class $\mathcal{R}(1, 1, 1, 0)$.

A structure of the minimal trees

For every $n \geq 4$, we define the class L_n as follows:

$$\mathcal{L}_n = \begin{cases} \mathcal{R}(2, k-1, 0, 0), \text{ if } n = 5k; \\ \mathcal{R}(0, k-1, 1, 0), \text{ if } n = 5k+1; \\ \mathcal{R}(1, k, 0, 0), \text{ if } n = 5k+2; \\ \mathcal{R}(3, k-1, 0, 0) \cup \mathcal{R}(0, k-1, 0, 1), \\ \text{ if } n = 5k+3; \\ \mathcal{R}(0, k+1, 0, 0), \text{ if } n = 5k+4. \end{cases}$$

Theorem 6.

For every $n \geq 4$, the class $\mathcal{L}_n$ consists of all minimal n-vertex trees.

- Clearly, for some n, the extremal trees are not unique.

Determining the lower bound

- Let $l(n)$ be the exact lower bound to the number of maximal i.s. in the class of n-vertex trees without twin-leaves.
- It is easy to see that for any tree $T \in \mathcal{R}(a, b, c, d)$ we have $mi(T) = 2^a \cdot 3^b \cdot 5^c \cdot 8^d$.
- Hence we have the following corollary from the previous theorem:

Corollary 7.

For every $n \geq 4$,

$$f(n) = \begin{cases} 4 \cdot 3^{k-1}, & \textit{if } n = 5k; \\ 5 \cdot 3^{k-1}, & \textit{if } n = 5k+1; \\ 6 \cdot 3^{k-1}, & \textit{if } n = 5k+2; \\ 8 \cdot 3^{k-1}, & \textit{if } n = 5k+3; \\ 9 \cdot 3^{k-1}, & \textit{if } n = 5k+4. \end{cases}$$

Conclusion

We consider two problems on counting maximal and maximum i.s. in different tree subclasses.

- First, for all n and d, we describe all the n-vertex trees of maximum vertex degree at most d.
  - For all even n, the extremal tree is unique, but for some odd n the uniqueness does not hold.
- Second, for all $n \geq 4$, we give the exact lower bound to the number of maximal independent set in the class of n-vertex trees.
  - We also describe all extremal trees, which are not unique for some n.

Thank you for your attention!