Multidimensional analogues of the Birkhoff and the König–Hall theorems for polyplexes

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Let $n, d \in \mathbb{N}$, and let $I_n^d = \{(\alpha_1, \ldots, \alpha_d) : \alpha_i \in \{1, \ldots, n\}\}$.
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$$\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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Let A_α be the d-dimensional submatrix of order $n-1$ obtained from A by deleting all hyperplanes containing index α.

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Complete 2-dimensional polyplex of order 4

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Incomplete 3-dimensional polyplex of order 3 and weight 2

$$\left(\begin{array}{ccc|ccc|ccc} 1/4 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{array}\right)$$

A polyplex K is primitive if there are no polyplexes of the same weight whose support is contained in the support of K.

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Non-primitive complete polyplex

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Primitive (but not diagonal) 3-dimensional polyplex of order 3

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Birkhoff thorem

Every doubly stochastic matrix is a convex combination of diagonal matrices.

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Theorem 1

If K is a complete primitive d-dimensional polyplex of order n then $n \leq |K| \leq dn - d + 1$.

Construction 1

For each $d \geq 3$ there exists a complete primitive d-dimensional polyplex of order 2 and with cardinality $dn - d + 1 = d + 1$.

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Hypergraph interpretation

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A fractional matching in a d-partite hypergraph is a polyplex in the d-dimensional adjacency matrix.

A fractional transversal in a hypergraph H is an assignment of nonnegative weights to vertices such that each hyperedge is covered with total weight at least 1.

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A hyperplane cover of a d-dimensional $(0,1)$-matrix A of order n is a nonnegative $(d \times n)$-table Λ such that for each α from the support of A it holds $\sum_{\Gamma_{i,j} \ni \alpha} \lambda_{i,j} \geq 1$. A weight of hyperplane cover Λ is $\sum_{i,j} \lambda_{i,j}$.

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A hyperplane cover Λ of a multidimensional $(0, 1)$-matrix A is exact iff each α from the support of A is covered with weight at least 1.

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Theorem

Let A be a multidimensional $(0,1)$-matrix, Λ be a hyperplane cover of A of a minimal weight, and K be a polyplex of a maximal weight in A.

1. If $k_\alpha > 0$ then $\sum\limits_{i=1}^{d} \lambda_{i,\alpha_i} = 1$.
2. If $\lambda_{i,j} > 0$ then $\sum\limits_{\alpha \in \Gamma_{i,j}} k_\alpha = 1$.

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König–Hall–Egerváry theorem

The maximal length of a positive partial diagonal in a 2-dimensional $(0, 1)$-matrix A is equal to the minimal number of lines covering all unity entries of A.

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Extremal 2-dimensional matrices of order n are exactly the matrices whose zero entries form a $(r \times t)$-rectangle with $r + t = n + 1$.

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Example: $n = 5$.

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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3-dimensional extremal matrices of order 2

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4. $\left(\begin{array}{ccc|ccc|ccc} 1 & \mathbf{1} & \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\Lambda = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$ $\delta = 1/2.$

3-dimensional extremal matrices of order 3

1. $$\left(\begin{array}{ccc|ccc|ccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \delta = 1.$$

2. $$\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \delta = 1.$$

3. $$\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix} \quad \delta = 1/2.$$

4. $$\left(\begin{array}{ccc|ccc|ccc} 1 & \mathbf{1} & \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{pmatrix} \quad \delta = 1/2.$$

5. $$\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix} \quad \delta = 1/3.$$

3-dimensional extremal matrices of order 3

1. $\left(\begin{array}{ccc|ccc|ccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \end{array}\right)$ $\quad \Lambda = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \delta = 1.$

2. $\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \delta = 1.$

3. $\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\quad \Lambda = \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$ $\quad \delta = 1/2.$

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5. $\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\quad \Lambda = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix}$ $\quad \delta = 1/3.$

6. $\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 1 & 1 & \mathbf{1} & 1 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 1 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\quad \Lambda = \begin{pmatrix} 3/4 & 1/2 & 0 \\ 1/2 & 1/4 & 0 \\ 1/2 & 1/4 & 0 \end{pmatrix}$ $\quad \delta = 1/4.$

Proposition 1

For deficiency δ of an extremal matrix it holds $0 < \delta \leq 1$.

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Proposition 2

Every extremal matrix is monotonic.

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Proposition 3

All hyperplane covers with a minimal weight of an extremal matrix are exact.

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All hyperplane covers with a minimal weight of an extremal matrix are exact.

Proposition 4

If Λ is a hyperplane cover with a minimal weight of an extremal matrix then each row of Λ contains at least one zero.

Proposition 5

1. There are no extremal matrices with deficiency $1/2 < \delta < 1$.
2. There are no extremal matrices with deficiency $1/3 < \delta < 1/2$.

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2. There are no extremal matrices with deficiency $1/3 < \delta < 1/2$.

Theorem 2

If A be a d-dimensional extremal matrix with deficiency δ equal to 1, $1/2$ or $1/3$ then the optimal hyperplane cover Λ is unique and all $\lambda_{i,j}$ are integer multiples of δ.

Theorem 3

1. There is a 1-to-1-correspondence between d-dimensional extremal matrices of order n and of deficiency 1 and the Young diagrams with $n-1$ cells and with no more than d rows.

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2. A $(d \times n)$-table Λ with $0, 1/2$ and 1-entries and with weight $n - 1/2$ is an exact hyperplane cover of an extremal matrix of order n with deficiency $\delta = 1/2$ if and only if the number of $1/2$-entries in each row of Λ is less than the number of $1/2$-entries in the union of all other rows.

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Theorem 4

If A is an extremal matrix with deficiency 1 or $1/2$ then for each $a_\alpha = 0$ the submatrix A_α contains a diagonal polyplex.

Conjecture 1

Every extremal matrix has the unique exact hyperplane cover of a minimal weight.

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Every extremal matrix has the unique exact hyperplane cover of a minimal weight.

Conjecture 2

1. The deficiency δ of any extremal matrix is $1/k$ for some $k \in \mathbb{N}$.

Conjecture 1

Every extremal matrix has the unique exact hyperplane cover of a minimal weight.

Conjecture 2

1. The deficiency δ of any extremal matrix is $1/k$ for some $k \in \mathbb{N}$.
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Every extremal matrix has the unique exact hyperplane cover of a minimal weight.

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1. The deficiency δ of any extremal matrix is $1/k$ for some $k \in \mathbb{N}$.
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Conjecture 3

1. If A is an extremal matrix then for all $a_\alpha = 0$ the submatrix A_α contains a complete polyplex.

Conjecture 1

Every extremal matrix has the unique exact hyperplane cover of a minimal weight.

Conjecture 2

1. The deficiency δ of any extremal matrix is $1/k$ for some $k \in \mathbb{N}$.
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Conjecture 3

1. If A is an extremal matrix then for all $a_\alpha = 0$ the submatrix A_α contains a complete polyplex.
2. If A is an extremal matrix then for all $a_\alpha = 0$ the submatrix A_α contains a diagonal polyplex.

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Every extremal matrix has the unique exact hyperplane cover of a minimal weight.

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Conjecture 3

1. If A is an extremal matrix then for all $a_\alpha = 0$ the submatrix A_α contains a complete polyplex.
2. If A is an extremal matrix then for all $a_\alpha = 0$ the submatrix A_α contains a diagonal polyplex.
   or
   If H is a d-partite hypergraph extremal for fractional matchings then adding a hyperedge to H produces a perfect matching.

Thank you for your attention!