Limit Topology for Dimer Algebra Random $\mathbb{R}$-Trees

(Based on Joint Discussion with Nicolai Reshetikhin)

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May 2018

Abstract

We prove inverse partition correlation of $\mathbb{Z}$ projective limit topology by Kasteleyn Pfaffian Pf$(\Sigma^K) \in \textbf{Quot}(\mathbb{K}[D])$ Grassmann field framework of bipartite $(\mathbb{Z}^+)^d$ dimer "tree-like object" equipped with dual lattice height. We obtain the asymptotics critical points of real and complex discriminants for the Grassmann-integral Fermion Kernel of special operators, and formulate conjecture for correlation function on large deviation functional of height function fluctuation φ, in asymptotic behavior of large random spanning tree and tree-valued measure.

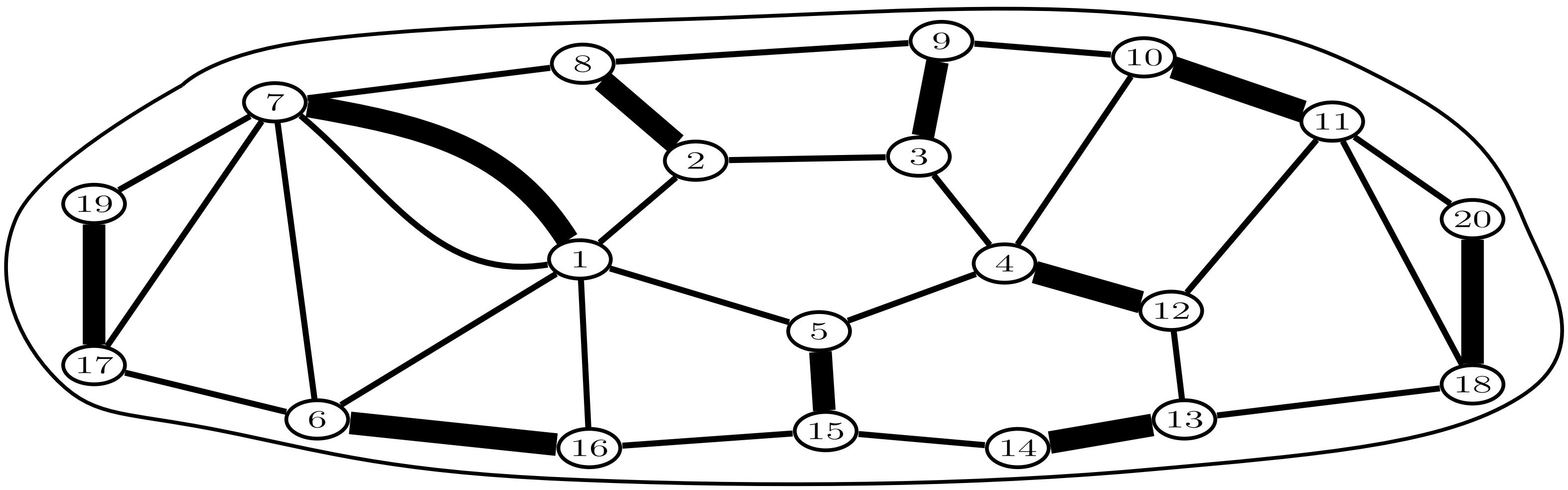
Keywords: Random-Dimer, Kasteleyn, Grassmann, Fermionic-Correlation

Grateful to all the conference organizers/participants

1 Characterizations

1.1 Basic definitions and observations

Graph $\Gamma := \{i \in \mathbb{N} \,|\, 1 \leq i \leq n\} \subset (\mathbb{Z}^+)^d \,|\, d = 2,3,\ldots$ in topological surface $\overline{\mathcal{M}}_g$



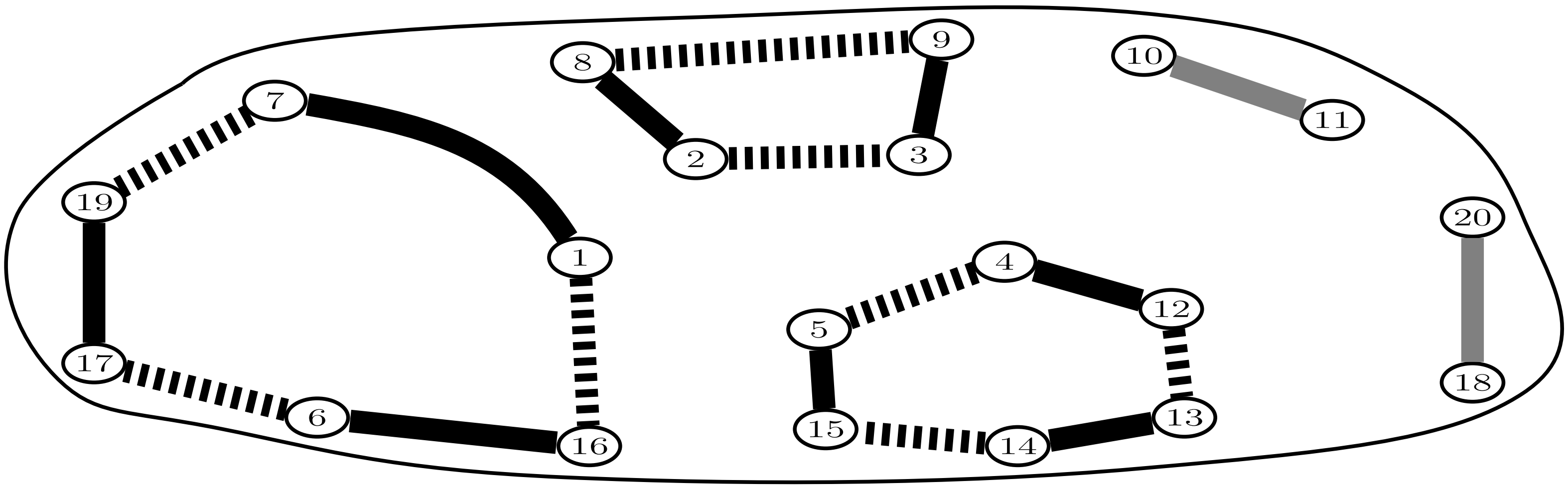
is partition $[\boldsymbol{\sigma}]$ if, $\forall\, \ell := \{i = i_1 \cdots i_d,\ j = j_1 \cdots j_d\} \in D$, $\exists$ perfect matching:

- Dimers $(\ell)_{\ell \in D}$ do not overlap, $\forall\, \mathscr{D} :=$ set of all configurations D of Γ
- All vertices $(i)_{i \in \Gamma} := \partial D$ are covered: $|\partial D| = 2 \sum_\ell \boldsymbol{\sigma}_D(\ell)$, $\boldsymbol{\sigma}_D := \{0, 1\}$

$\forall\, |[\boldsymbol{\sigma}]| \leq |\mathscr{D}|$ partitions $[\boldsymbol{\sigma}]$.

Remark. $\Gamma :=$ closed, connected; $n :=$ even; dimer is undefined $\forall\, i = j$ loop.

Symmetric difference $D\Delta D' := D \cup D' \setminus D \cap D' \supseteq (C_\alpha \mid \alpha \in \mathbb{N}, 1 \leq \alpha \leq m)$.



One or more circuit-coverings $\Longleftarrow$ transition cycles := superposition cycles:

- Disjoint even-length (simple closed curves) cycles $C_\alpha \mid \alpha \in \mathbb{N}, 1 \leq \alpha \leq m$.
- Alternating sequence of configuration-pair by ordered cyclic sequence: elementary circuit := $(\ell_1, \ell_2, \ldots, \ell_r)$ of $(\ell_1, \ell_3, \ldots, \ell_{r-3}, \ell_{r-1}) \in D$, resp. $(\ell_2, \ell_4, \ldots, \ell_{r-2}, \ell_r) \in D'$, for sequence $i_1, \ell_1, i_2, \ell_2, \ldots, \ell_r$ joining vertex i_1 after finite edges, starting at i_1. Another elementary circuit starting at vertex i_{r+1} at end of edge $\ell_{r+1} \in D$ for Γ of vertices $> r$, etc.

Proposition. $\mathsf{Aut}(D)$:=subset of automorphism group $\mathsf{Aut}(\mathscr{D})$ is given by

$$|\mathsf{Aut}(D)| = \left(\frac{n}{2}\right)!\; 2^{\frac{n}{2}}, \text{ where } |\mathsf{Aut}(\mathscr{D})| \le |\mathsf{Aut}(\Gamma)| = n!\,.$$

Proof. Preserving contiguity (adjacency + distribution), $\forall\,[\boldsymbol{\sigma}] \sim \boldsymbol{\sigma} \in \mathsf{Aut}(D)$:

$$[\boldsymbol{\sigma}] : \Gamma \to \Gamma \begin{cases} \boldsymbol{\sigma}(1,\ldots,n) = (\boldsymbol{\sigma}(1), \boldsymbol{\sigma}(2), \ldots, \boldsymbol{\sigma}(n-1), \boldsymbol{\sigma}(n)) \\ \text{s.t. } \boldsymbol{\sigma}(2\ell-1) = \ell,\ \ell < \boldsymbol{\sigma}(2\ell),\ \forall\,\ell = 1,\ldots,\frac{n}{2}, \end{cases}$$

and

$$\mathsf{Aut}(\mathscr{D}) = \begin{cases} \boldsymbol{\sigma} = [\boldsymbol{\sigma}] \\ \forall\,\mathcal{S}_{\frac{n}{2}}, \mathcal{S}_2^{\frac{n}{2}} \end{cases} \left| \begin{array}{l} \mathcal{S}_{\frac{n}{2}} := (1,\boldsymbol{\sigma}(1),\ldots,\frac{n}{2},\boldsymbol{\sigma}(\frac{n}{2})),\ldots,(\frac{n}{2},\boldsymbol{\sigma}(\frac{n}{2}),\ldots,1,\boldsymbol{\sigma}(1)) \\ \mathcal{S}_2^{\frac{n}{2}} := (1,\boldsymbol{\sigma}(1),\ldots,\frac{n}{2},\boldsymbol{\sigma}(\frac{n}{2})),\ldots,(\boldsymbol{\sigma}(1),1,\ldots,\boldsymbol{\sigma}(\frac{n}{2}),\frac{n}{2}) \end{array} \right.$$

That is, $|\mathsf{Aut}(D)| = |\mathsf{Aut}(\mathscr{D})| / |[\boldsymbol{\sigma}]|$ by $[\boldsymbol{\sigma}]$ well-defined. $\square$

Remark. Given minimum distinct valency k of the graph Γ

$$k \le |[\boldsymbol{\sigma}]| \le n! \Big/ \left(\frac{n}{2}\right)!\, 2^{\frac{n}{2}} = (n-1)!! = (n-1)\cdots 3\cdot 1 =: |[\boldsymbol{\sigma}]| \text{ for } k = (n-1).$$

Remark. All Γ configurations, i.e. of all partitions, is given by

$$\sum_{N_1,\ldots,N_d} \sum_{D(N_\nu)} 1 \;\left|\; \sum_{\nu=1}^{d} N_\nu = \frac{n}{2},\ D := N_\nu \text{ edges of classes } C_\nu,\ \forall\,\nu = 1,\ldots,d. \right.$$

The local observables := D-D correlation functions (conditional probabilities)

$$\left\langle \prod_{\ell \in D} \sigma_D(\ell) \right\rangle \stackrel{\text{def}}{=} \text{Prob}(\ell_1 \in D, \ldots, \ell_n \in D) := \mathbb{E}\left[\prod_{i=1}^{n} \sigma_D(\ell_i)\right]$$

$$= \sum_{D \in \mathscr{D}} \prod_{i=1}^{n} \sigma_D(\ell_i) \times \text{Prob}(D) = \frac{\sum_{D \in \mathscr{D}} \prod_{\ell \in D} \varepsilon_\ell \, \omega_\ell \prod_{i=1}^{n} \sigma_D(\ell_i)}{\sum_{D \in \mathscr{D}} \prod_{\ell \in D} \varepsilon_\ell \, \omega_\ell}$$

where $\varepsilon_\ell \in \{-1, +1\}$, and $\sigma_D : \Gamma \to \{0, 1\}$ by

$$\sigma_D(\ell) := \begin{cases} 1 & \text{if} \quad \ell \in D \\ 0 & \text{if} \quad \ell \notin D \end{cases} \qquad \forall\, D \in \mathscr{D}$$

is the indicator function of ℓ on the support $\mathscr{D}$ for all dimer weights $\omega_{(\cdot)}$

and,

$$\text{the numerator} = \sum_{D \ni \ell_1, \ldots, \ell_n} \varepsilon_D \, \omega_D \quad \Big| \quad \varepsilon_D \in \{-1, +1\} \sim \prod_{\ell \in D} \varepsilon_\ell \, .$$

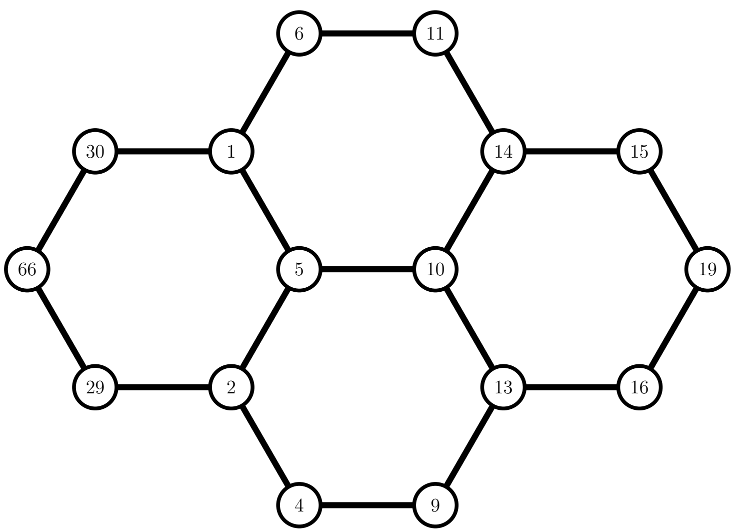
Remark. If any $\ell_\xi \neq \ell_\eta \mid \xi, \eta \in \{1, \ldots, n\}$ have a common vertex, then $\langle \sigma_D(\ell_1) \cdots \sigma_D(\ell_n) \rangle = 0$; more so, $\sigma_D(\ell)\, \sigma_D(\ell) = \sigma_D(\ell)$; so, $\ell_\xi \neq \ell_\eta$, $\forall\, \xi \neq \eta$.

That is, in the σ-finite energy $\Xi_{(\cdot)}$ measure $\delta(\cdot) \in \Xi : E(\Gamma) \to \mathbb{R}$, $\ell \mapsto \Xi_\ell$, probability exists by

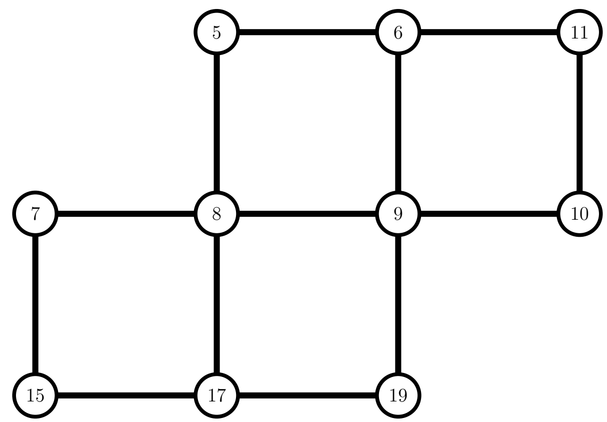
$$\left\langle \prod_{\ell \in D} \sigma_D(\ell) \right\rangle := \frac{\prod_{\ell \in D} \varepsilon_\ell \, \omega_\ell}{\sum_{D \in \mathscr{D}} \prod_{\ell \in D} \varepsilon_\ell \, \omega_\ell} = \frac{\varepsilon_D \, \omega_D}{Z(\Gamma; \omega)} =: \mathsf{Prob}(D) \Bigg|\, \omega_\ell = e^{-\frac{\Xi_\ell}{kT}},\ \varepsilon_D \sim \prod_{\ell \in D} \varepsilon_\ell$$

$$= \frac{1}{Z} \exp\left(-\frac{\Xi_D}{kT}\right) \quad \Bigg|\ \Xi_D = \sum_{\ell \in D} \Xi_\ell\,,\ \ Z = \sum_{D \in \mathscr{D}} \exp\left(-\frac{\Xi_D}{kT}\right)$$

where $Z :=$ strict-sense positive continuous partition function on the objects:



- Domains in hexagonal lattice.

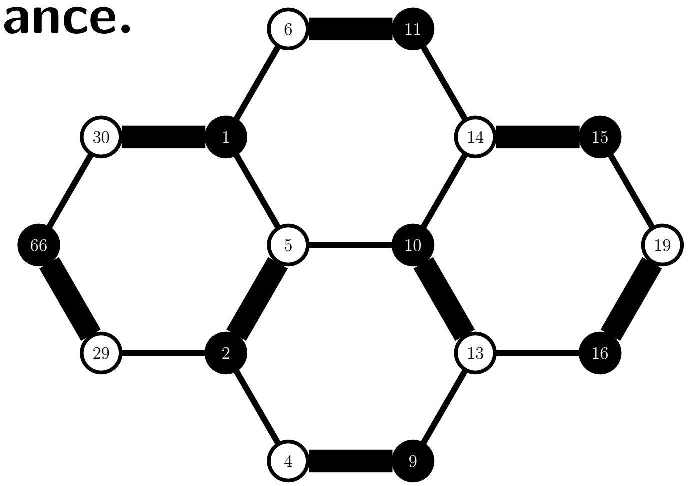


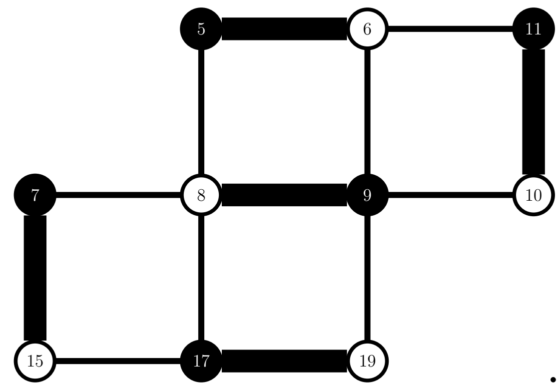
- Domains in square lattice.

The *bipartite* graph (of no black-black nor white-white edge) is disjoint union

$$V(\Gamma) = V_\bullet(\Gamma) \sqcup V_\circ(\Gamma).$$

Instance.





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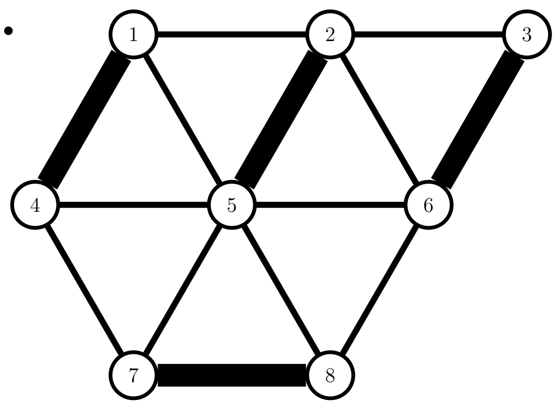
Non-instance.



$\left(\begin{array}{l}\textit{no bipartite structure}\\ \textit{on triangular lattices}\end{array}\right)$.

Remark.

$D :=$ 1-chain in cell complex of GF_2 field: $(\ell)_{\ell \in D} \in \mathcal{H}^1\big(\overline{\mathcal{M}}_g; GF_2\big) \cong (\mathbb{Z}_2)^{2g}$.

$\partial D :=$ 0-chain in cell complex of GF_2 field: $(i)_{i \in \Gamma} \in \mathcal{H}^0\big(\overline{\mathcal{M}}_g; GF_2\big) \cong \mathbb{Z}_2$.

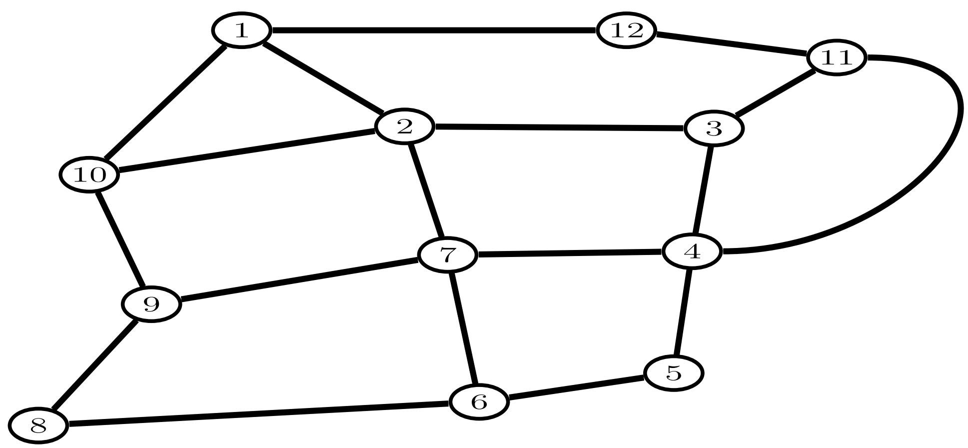
1.2 Combinatorial equivalence

Proposition. *Two families, respectively, on space of dimers and space of tilings, are combinatorially equivalent; that is,* $\exists$ *bijective correspondence*

$$\boldsymbol{Dimers} \;\longleftrightarrow\; \boldsymbol{Tilings}.$$

Proof. Let all $(v)_{v\,\in\,\Gamma}\subset\overline{\mathcal{M}}_g :=$ *closed* traversal (walk); Γ finite, connected, and planar (non-intersecting edges). The set of all Γ spanning trees defines:

I. The 2D cell complex Γ:
vertices, edges, faces $:=$ 0-cells, 1-cells, 2-cells, resp.



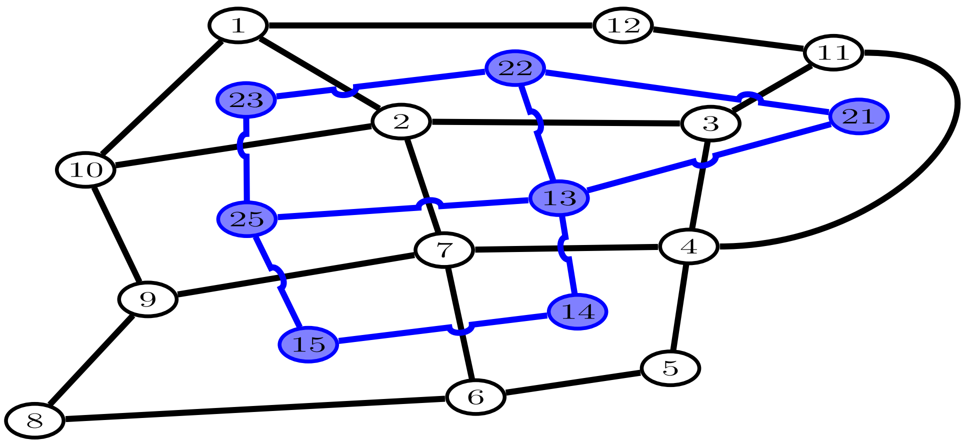
Disjoint interiors.

$$\dim(\partial\Gamma_n) = (n-1)\,/GF(2)$$

$\partial\Gamma_n :=$ *boundary between two* n*-cells.*

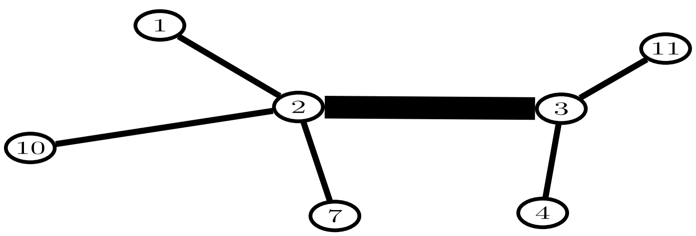
Remark. $\Gamma\subset\overline{\mathcal{M}}_g :=$ embedding by 1-skeleton CW-complex (resp. cellular decomposition of an oriented closed connected surface).

II. The dual cell complex Γ^*:
0-cells, 1-cells, 2-cells $:=$ "*centers*" of 2-cells, 1-cells, 0-cells of Γ, resp.

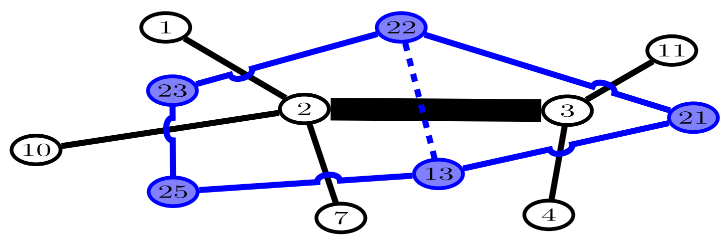


$\Gamma^* :=$ dual cell complex to Γ.

III. Dimer on Γ:



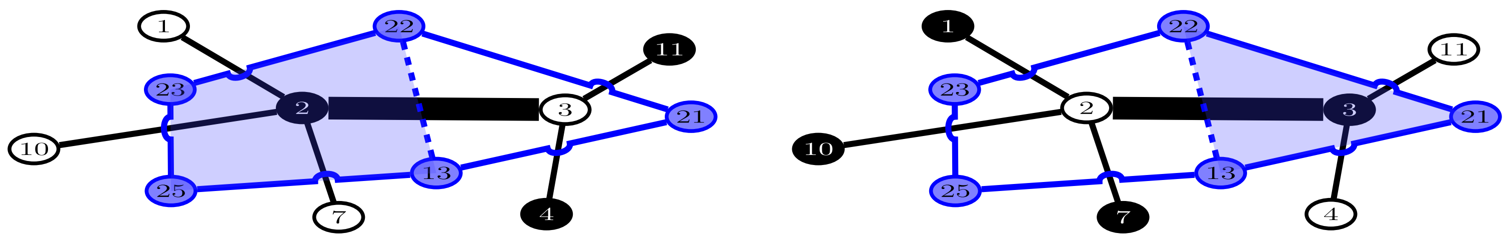
Unique pair of 2-cells on Γ^* share a dimer:



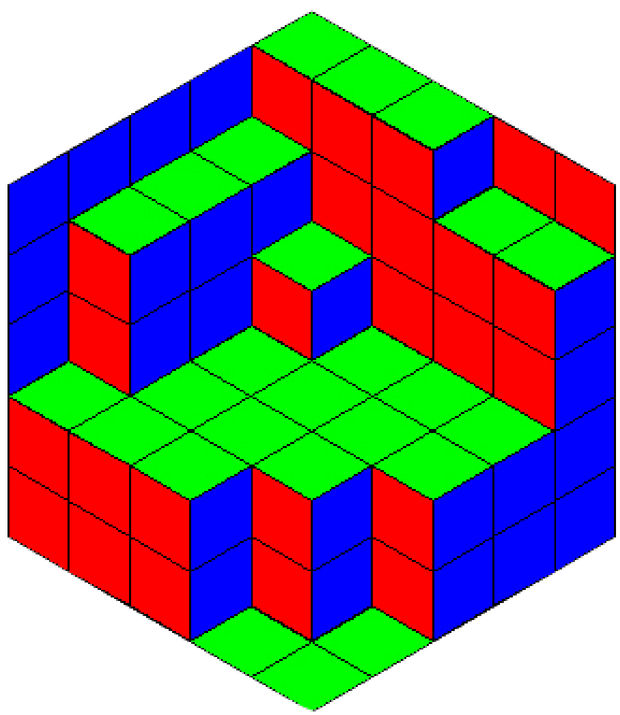
IV. Therefore, the global bijection:

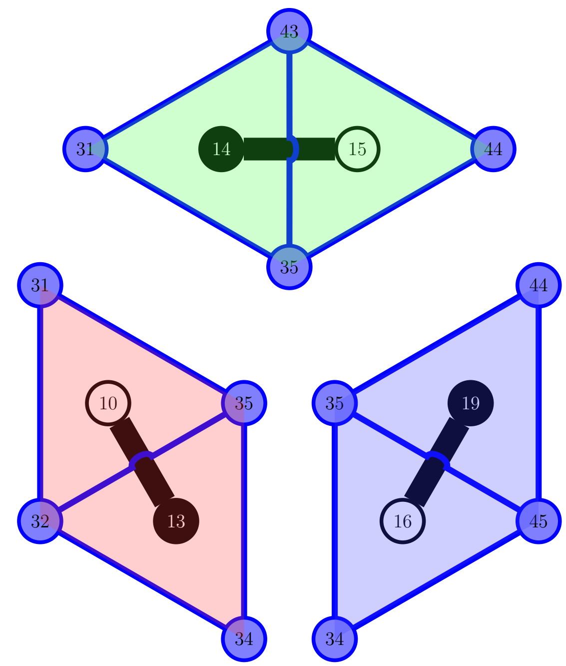
$$(\text{Dimers on } \Gamma) \longleftrightarrow \left(\begin{array}{l} \text{Tilings of } \Gamma^* \text{ by} \\ \textit{unique pair} \text{ of } 2\text{-cells} \end{array} \right).$$

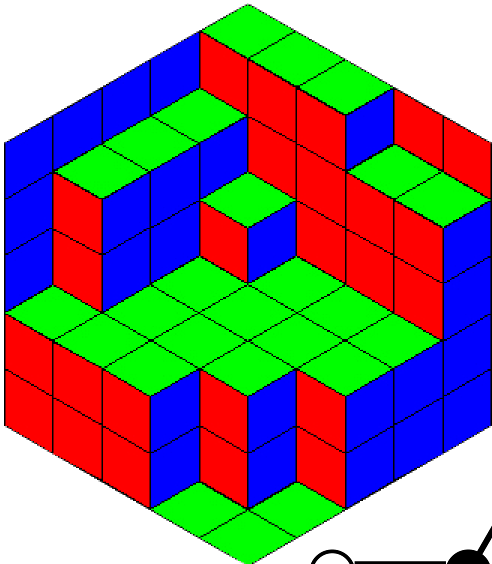
In particular, on the dual cell complex of bipartite graph:



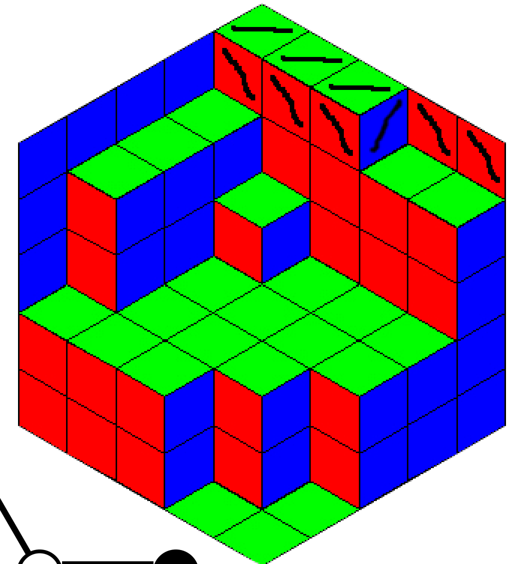
(two "colorings" of the same tile).

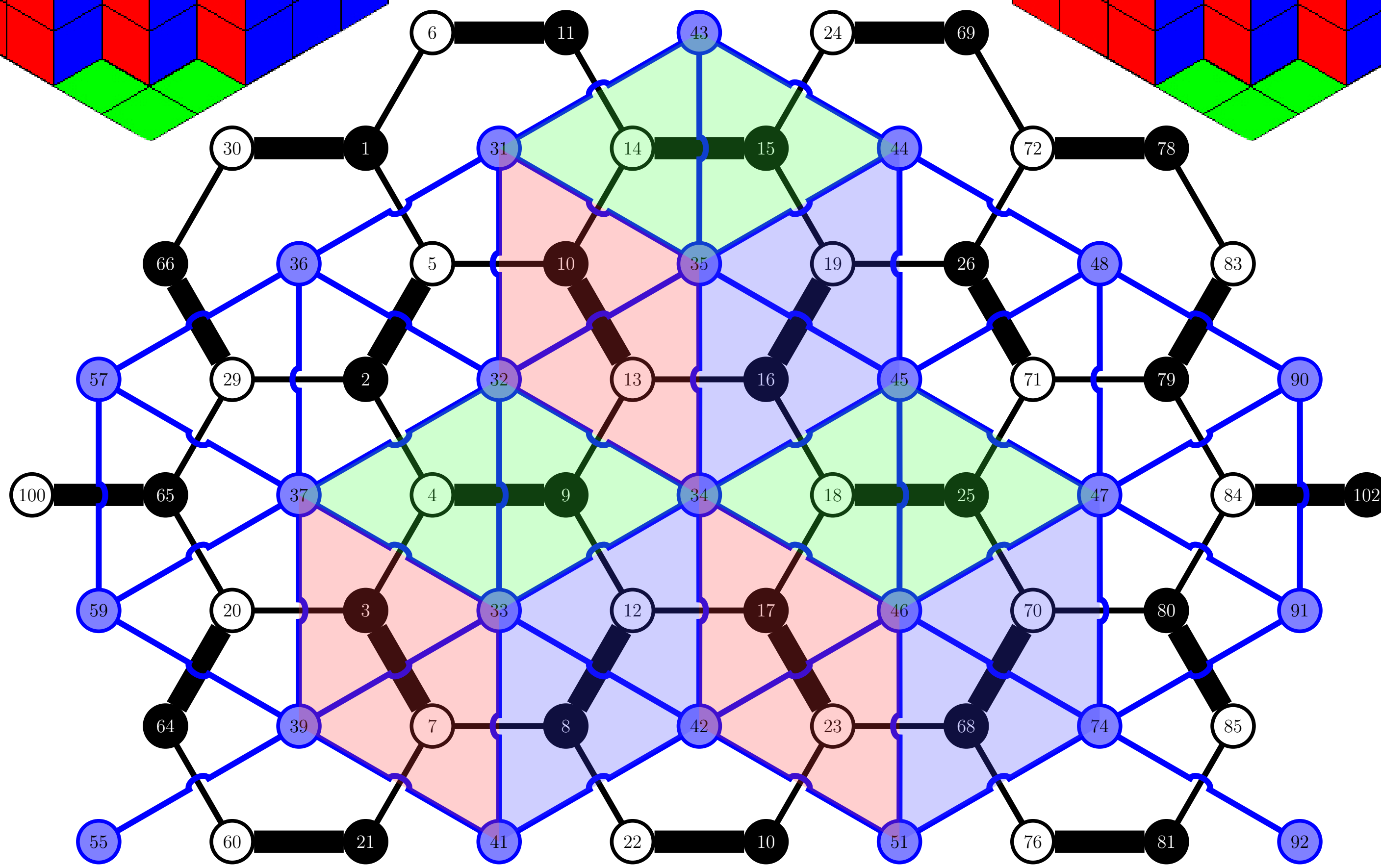






Limit topology := stacked 3D boxes; projection of 2D rhombus tiling:





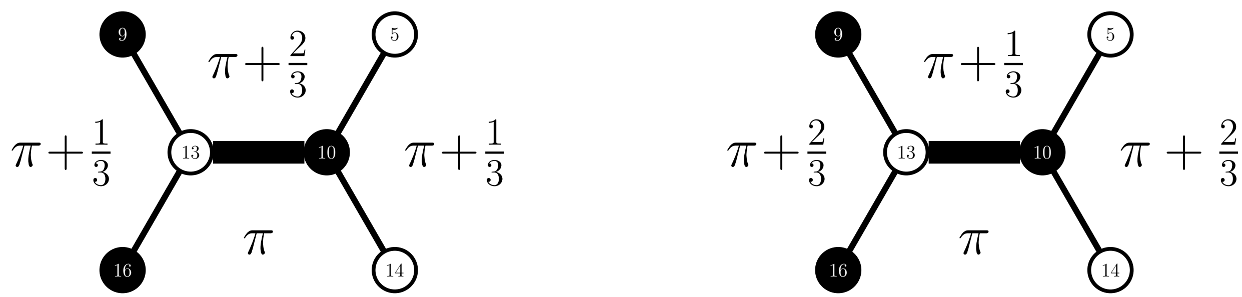
1.3 Space of height functions

Definition. Let $\Gamma \subset \overline{\mathcal{M}}_g :=$ planar, bipartite, hexagonal, such that

$$\textbf{Dimers} \longleftarrow \textbf{Discrete Surfaces.}$$

The space of height functions is the partially ordered by overlap, given by:

$$\mathcal{H}_\Gamma \overset{\text{def}}{=\!=} \{\pi : \text{faces}\,(\Gamma) \to \mathbb{Z}\}$$



with respect to normalization $\pi(f_0) = 0$ of the reference face f_0.

Proposition.

(i). Let $\partial\Gamma :=$ boundary faces. Then, $\pi_D\big|_{\partial\Gamma}$ does not depend on D.

(ii). $\pi_{D_1 D_2} = \pi_{D_1} - \pi_{D_2}$.

Proof. ♡.

Theorem. *The dimer configuration probability is given by*

$$\mathrm{Prob}(D) = \frac{1}{Z}\prod_f q_f^{\pi_D(f)} \;\Big|\; Z = \sum_{\pi\in\mathcal{H}_\Gamma}\prod_f q_f^{\pi(f)},\;\; q_f = \prod_{\ell\in\partial f}\omega_\ell^{\varepsilon_\ell},\;\; \pi\in\mathcal{H}_\Gamma,\; D\cong\pi.$$

Proof. By the combinatorial equivalence,

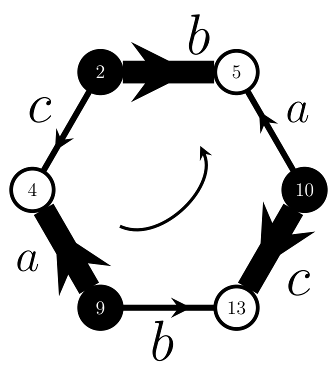
$$\Big\{\textit{Dimers on }\Gamma\Big\} \underset{\textit{bijection}}{\cong} \Big\{\textit{height functions}\Big\}.$$

The bijection then gives the measure. □

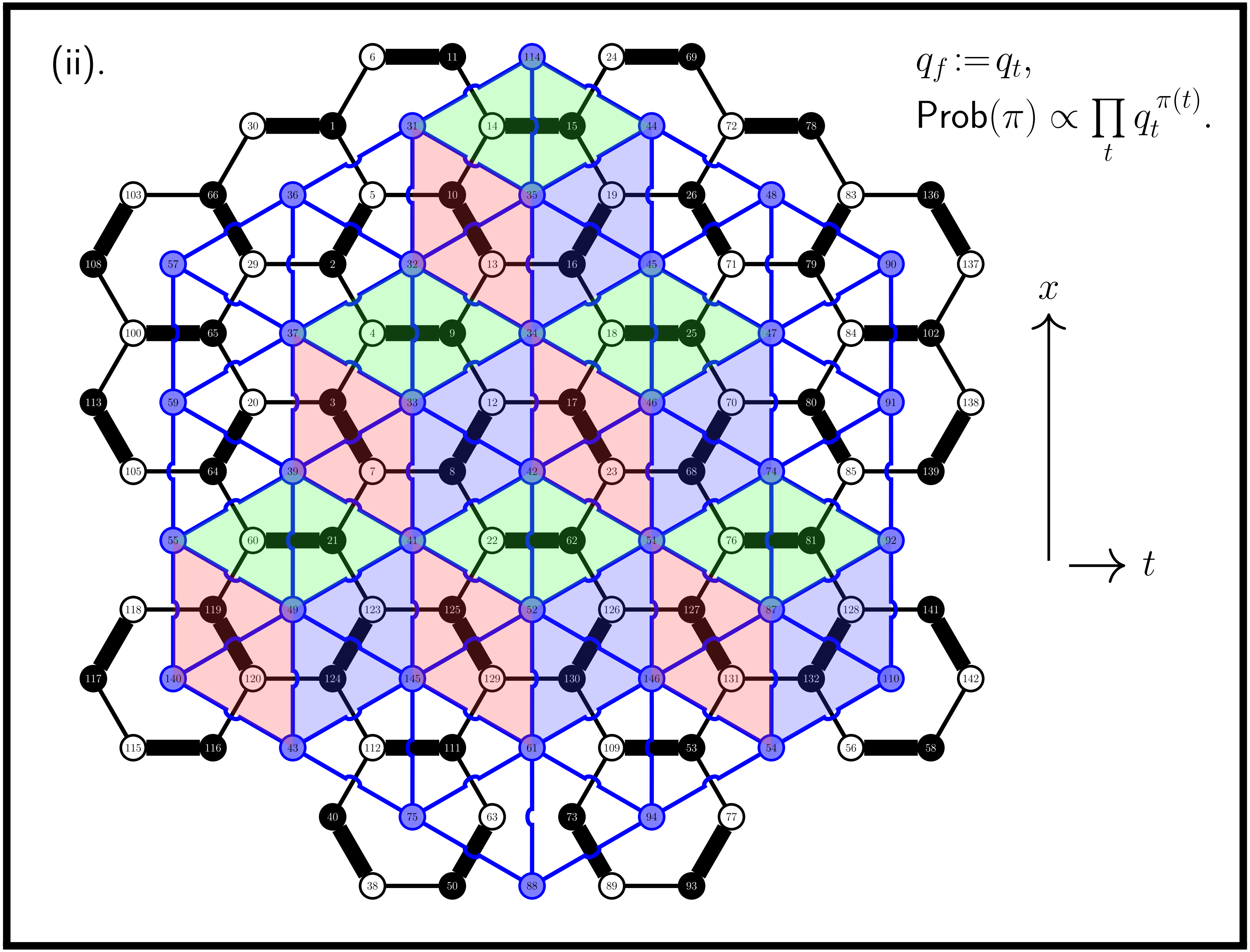
Remark. $\mathrm{Prob}(D)$:= "gauge" invariant: $\omega(\ell) \longmapsto s(\ell_+)\,\omega(\ell)\,s(\ell_-)$. More so, q_f := invariant ("essential" parameters).

Particular cases.

(i).



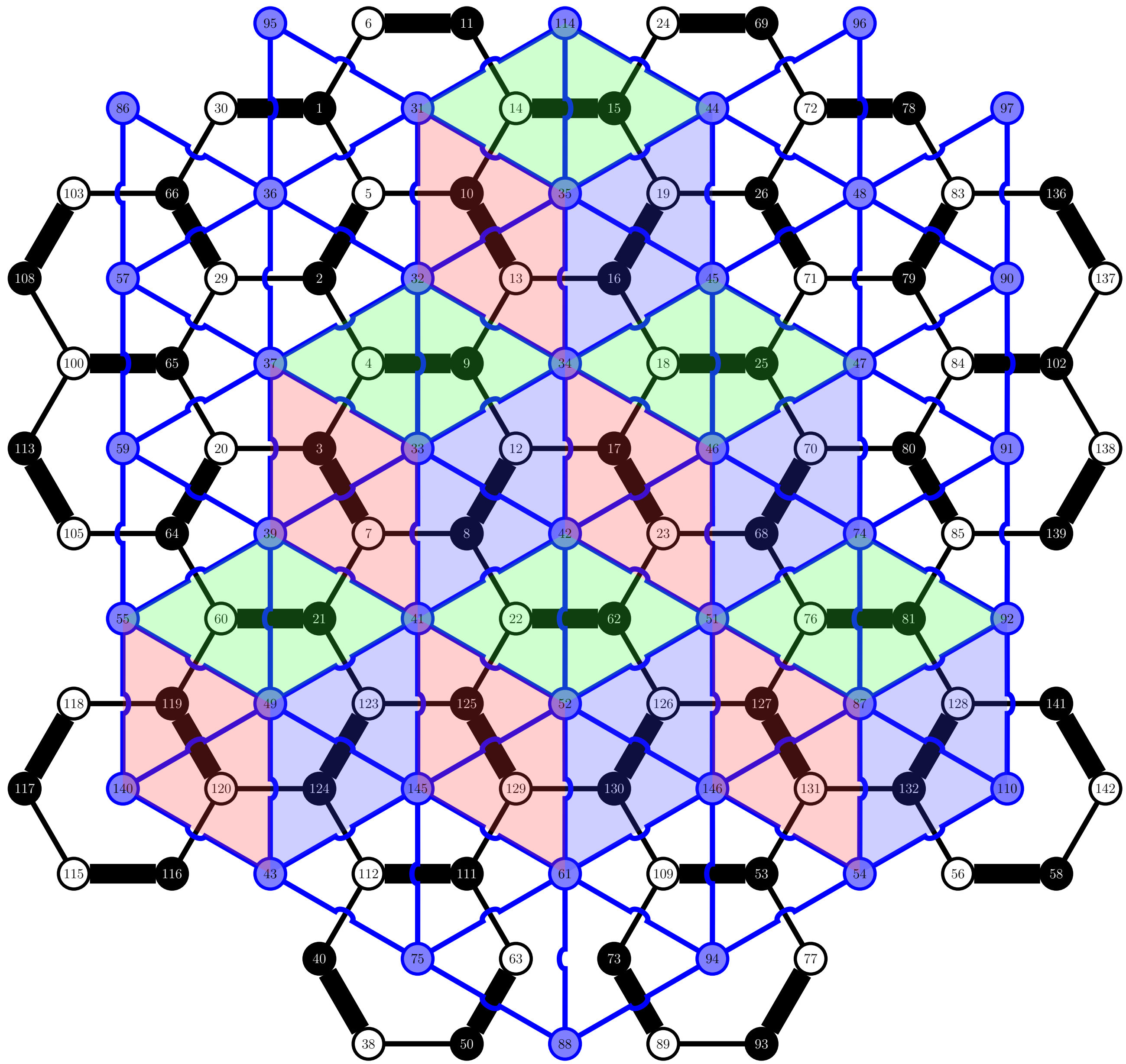
$\Longrightarrow$ the uniform distribution:
$q = a^{-1}b\,c^{-1}a\,b^{-1}c = 1.$



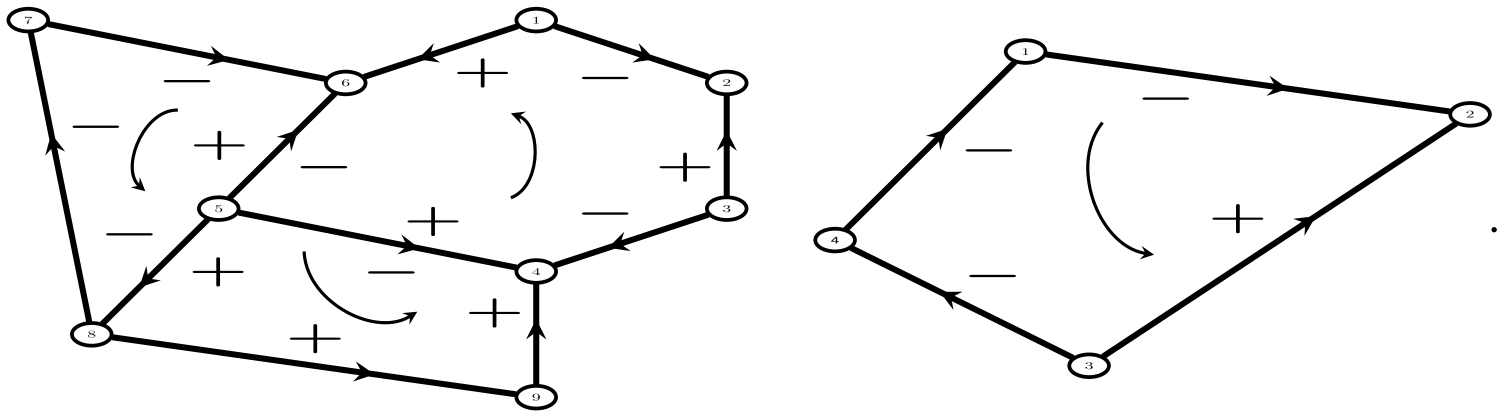
(ii).
$q_f := q_t,$
$\mathsf{Prob}(\pi) \propto \prod_t q_t^{\pi(t)}.$
x
t

Remark.
$0 < q_t < 1$
gives
unbounded
stacks:

$\text{Prob}(\pi)$
$\propto \prod_t q_t^{\pi(t)}$.



1.4 Kasteleyn orientation and matrix



Definition. Let $\Gamma\subset(\mathbb{Z}^+)^d :=$ CW-complex 1-skeleton (resp. genus g compact orientable surface cell-decomposition) closed connected embedding, endowed with induced (counterclockwise) orientation $\varepsilon(\partial\mathcal{F})$ on every face boundary; Γ is Kasteleyn if $\forall\{i,j\}$ edges ℓ, $i\neq j$, arbitrary i-to-j orientations $\varepsilon^K(i_\ell j_\ell)$,

$$\prod_{\ell\,\in\,\partial\mathcal{F}}\varepsilon^K_{i_\ell j_\ell}=-1,\ \forall\,\mathcal{F}\in\Gamma\ \ \Big|\ \varepsilon^K_{i_\ell j_\ell}:=\begin{cases}-1 & \text{if } \varepsilon(\partial\mathcal{F})\in\text{counter-}\varepsilon^K(i_\ell j_\ell)\\ +1 & \text{if } \varepsilon(\partial\mathcal{F})\in\varepsilon^K(i_\ell j_\ell).\end{cases}$$

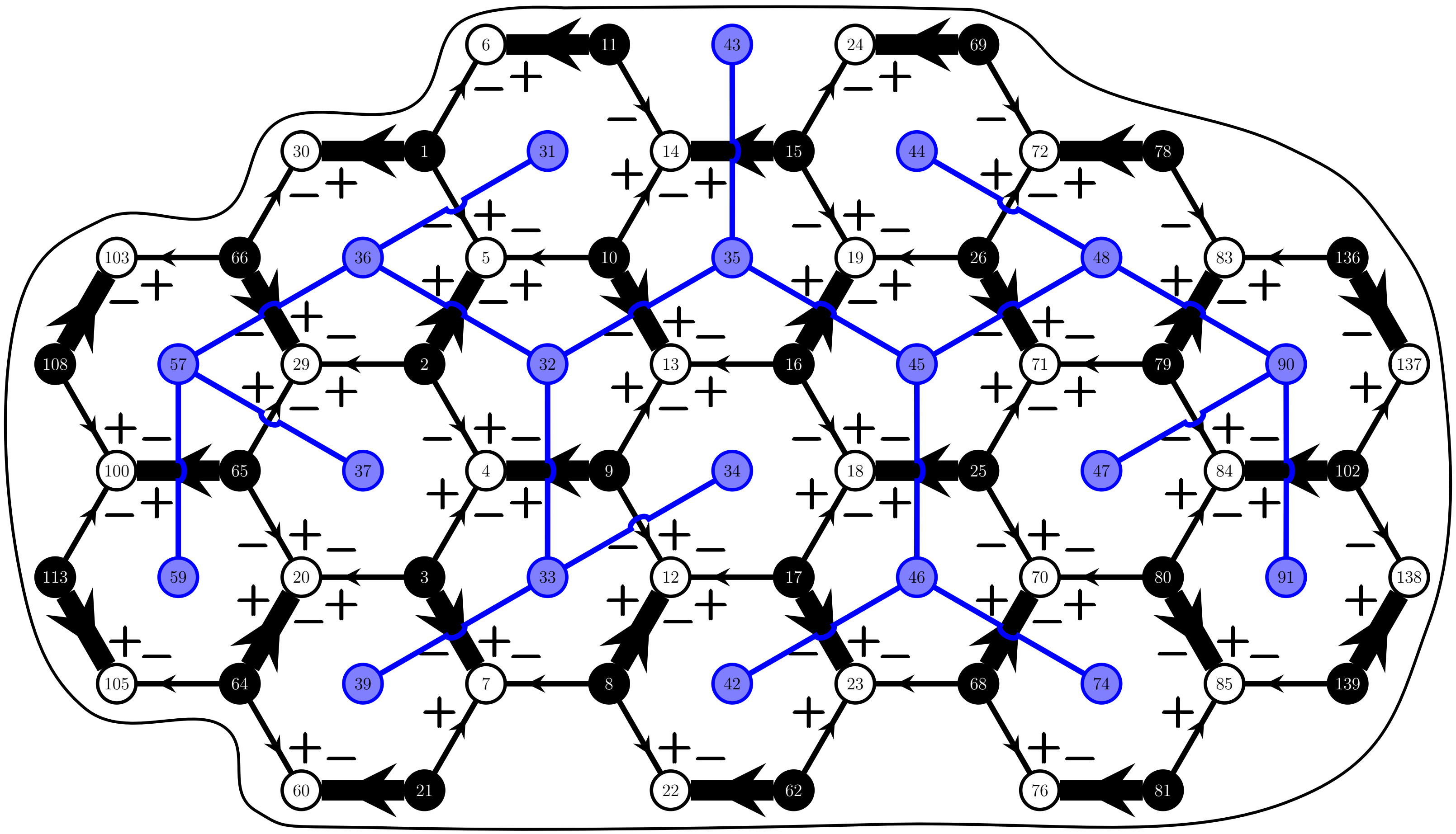
Let $\Gamma :=$ Kasteleyn, weighted $\omega>0$, $\forall\{i,j\}$ edges ℓ, $i\neq j$,

$$\Sigma^K_{ij}=\sum_\ell\varepsilon^K_{i_\ell j_\ell}\,\omega_{i_\ell j_\ell}\ \Big|\ \Sigma^K_{ij}:=0\ \text{ for } i,j \text{ not adjoined and } i=j \text{ loop.}$$

Proposition (existence). *Kasteleyn orientation exists.*

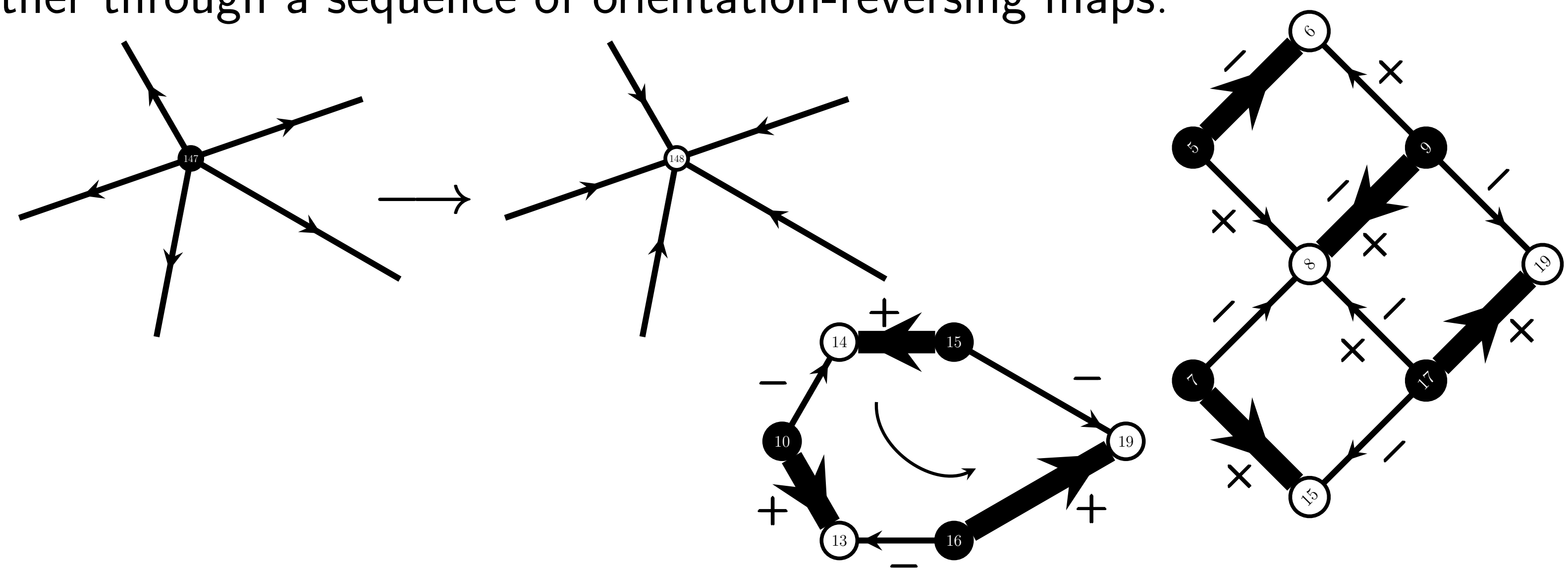
Proof. ♡.

Construction (bipartite case). Following Γ^* spanning tree:



Reduce Γ to small $N \times N \to \exp(\alpha N^2)$; choose $\varepsilon(\partial \mathcal{F})$; follow Γ^* spanning tree, rooted outside Γ, from root; make $\varepsilon^K(\mathcal{F})$ with each Γ^* edge deletion.

Definition. Two orientations are equivalent, if each is obtainable from the other through a sequence of orientation-reversing maps:



Theorem. *All Kasteleyn orientations of Γ planar are equivalent.*

Proof. Induction, $\forall\, n < \infty$ Kasteleyn orientations. $\square$

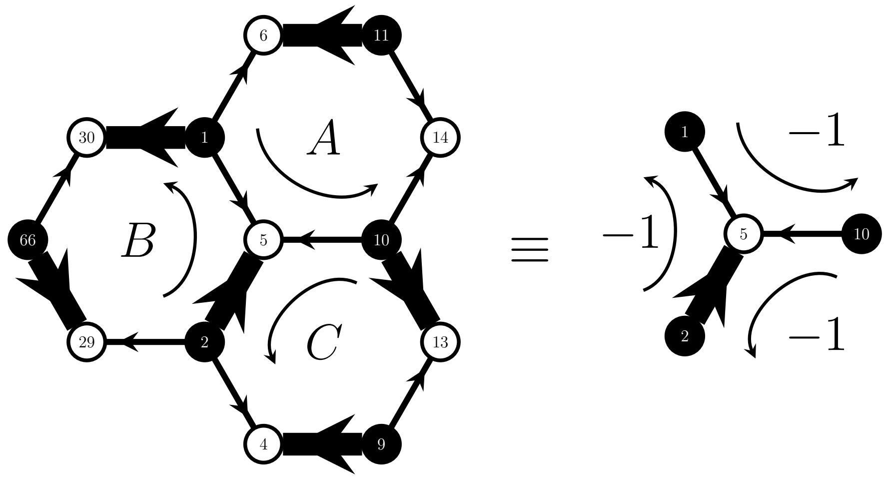
Lemma (generalization of R., et. al., 2008). *The set of equivalence classes of K-orientations on $\Gamma \subset \overline{\mathcal{M}}_g$ is an affine space* $:= \mathcal{H}^1(\overline{\mathcal{M}}_g; GF_2)$.

Proof. $\heartsuit$.

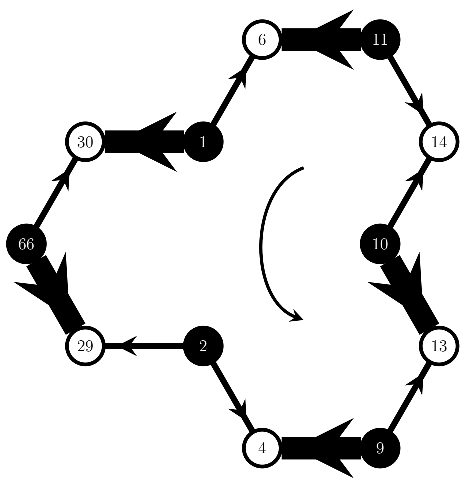
Corollary. $\Gamma \subset \overline{\mathcal{M}}_g :=$ exactly 2^{2g} equivalence classes of K-orientations, where $g :=$ genus of the topological surface $\overline{\mathcal{M}}_g$.

Proof. $\heartsuit$.

Remark. Deleting a vertex changes orientation to non-Kasteleyn at “holes”:



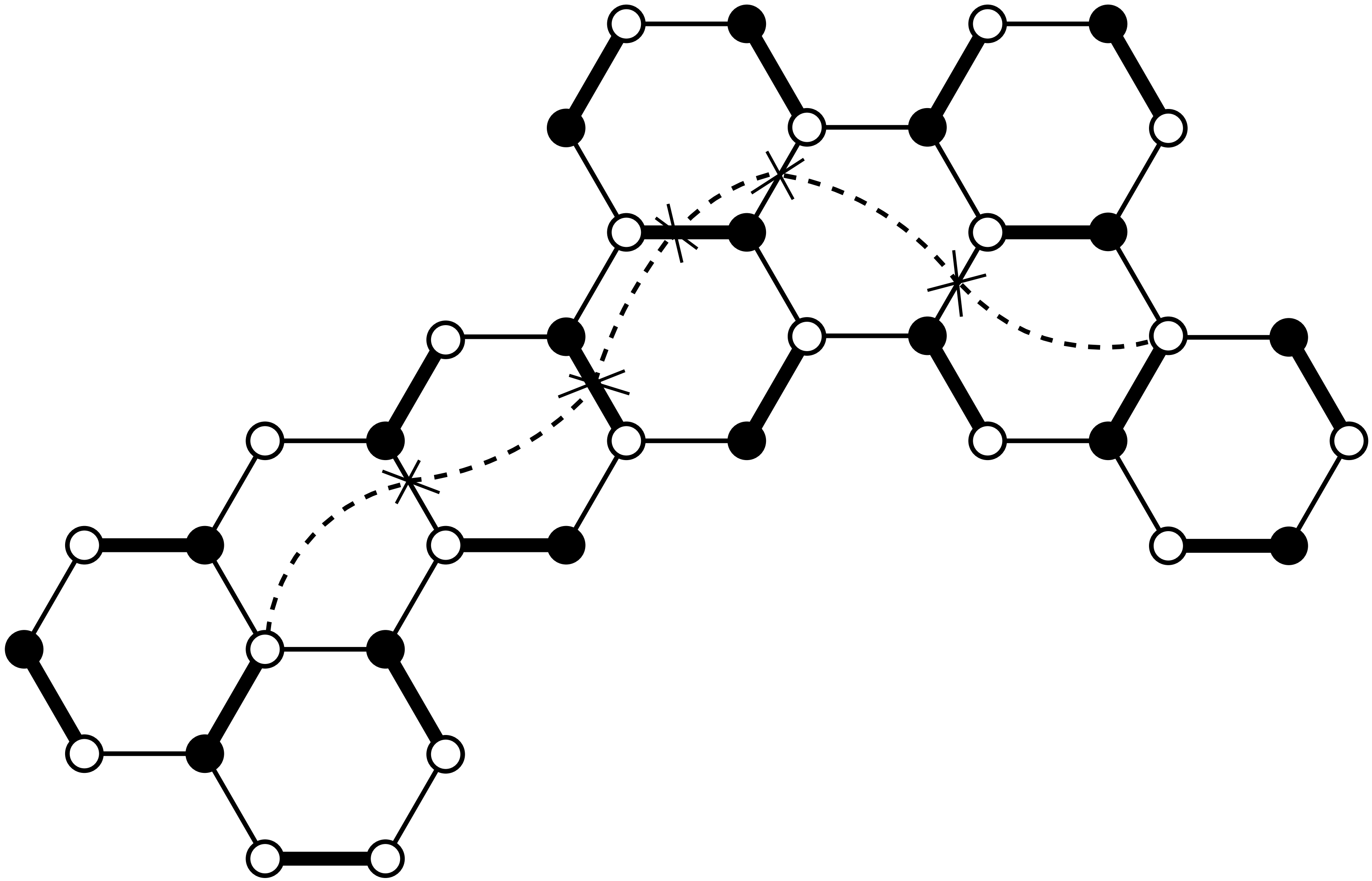
$h_A h_B h_C = (-1)^3 = -1$



$h_{ABC} = 1$, non-Kasteleyn orientation.

That is, $h_A h_B h_C = -h_{ABC}$.

Remark. To convert the non-Kasteleyn orientation back to Kasteleyn:



$h_A h_B h_C = 1.$

Derivation. Given $\Gamma \subset (\mathbb{Z}^+)^d :=$ cycle-graph in plane, $\forall\, i, j = 1, \ldots, n$

$$\Sigma^K_{ij} = \begin{cases} -\omega_{ij} = \omega_{ji} & \text{for } \varepsilon(\partial\mathcal{F}) \in \text{counter-}\varepsilon^K_{ij},\ \forall\, i \bullet\!\!\Longrightarrow\!\!\circ\, j,\ i \bullet\!\!\longrightarrow\!\!\circ\, j \\ \omega_{ij} = -\omega_{ji} & \text{for } \varepsilon(\partial\mathcal{F}) \in \text{counter-}\varepsilon^K_{ij},\ \forall\, i \circ\!\!\Longleftarrow\!\!\bullet\, j,\ i \circ\!\!\longleftarrow\!\!\bullet\, j \\ 0 & \text{for } i, j \text{ not adjoined and loop } i = j. \end{cases}$$

Theorem (Kasteleyn). *Every* $\pm|\det \Sigma^K|^{\frac{1}{2}} =: \mathrm{Pf}(\Sigma^K) \in \mathbf{Quot}(\mathbb{K}[D])$, $\Gamma \subset (\mathbb{Z}^+)^d$, *non-vanishing monomial corresponds* 1-1 *to* $[\sigma]$, $\forall\, D$ *on* N_ν *edges of class* $C_\nu \mid \nu = 1, \ldots, d$, *with respect to sgn*$(\sigma) := (-1)^{t(\sigma)}$ *where* $t(\sigma) := (\sigma(i \in \mathbb{N}) \mid 1 \leq i \leq n) \longrightarrow (i \in \mathbb{N} \mid 1 \leq i \leq n)$, $\forall\, \sigma \in$ *Aut*(D), *such that*:

(i) $$|\mathrm{Pf}(\Sigma^K)| = \sum_{D \in \mathscr{D}} \prod_{\ell \in D} \varepsilon_\ell\, \omega_\ell \;\Big|\; \mathrm{Pf}(\Sigma^K) = \sum_{\sigma = [\sigma]} (-1)^{t(\sigma)} \prod_{\ell=1}^{\frac{n}{2}} \Sigma^K_{\sigma(2\ell-1)\sigma(2\ell)}.$$

(ii) $$\mathrm{Pf}(\Sigma^K) = \frac{1}{2^{\frac{n}{2}}} \frac{1}{\left(\frac{n}{2}\right)!} \sum_{\sigma} (-1)^{t(\sigma)} \prod_{\ell=1}^{\frac{n}{2}} \Sigma^K_{\sigma(2\ell-1)\sigma(2\ell)} =: \sum_{N_1, \ldots, N_d} \sum_{D} \prod_{\nu=1}^{d} (\pm)\omega_\nu^{N_\nu}.$$

Proof. (ii) follows from (i) by Aut(D).

To see (i): $\det \Sigma^K = \det(-\big(\Sigma^K\big)^T) = \det(-\Sigma^K) = (-1)^n \det \Sigma^K$ implies $\det \Sigma^K$:= strictly positive-definite square of a rational function of array Σ^K_{ij}, $\forall\, i, j = 1, \ldots, n$, n := even, on the Leibniz's second-index permutations:

$$\sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} (-1)^{t(\sigma)} \Sigma^K_{i\,\sigma(i)} = \sum_{\sigma=1}^{n!/2} \left(\prod_{\substack{i=1 \\ \sigma(i) := \text{given by} \\ \text{even permutations} \\ \sigma \in \mathcal{S}_n}}^{n} (-1)^{t(\sigma)} \Sigma^K_{i\,\sigma(i)} + \prod_{\substack{i=1 \\ \sigma(i) := \text{given by} \\ \text{odd permutations} \\ \sigma \in \mathcal{S}_n}}^{n} (-1)^{t(\sigma)} \Sigma^K_{i\,\sigma(i)} \right)$$

where

$$\sigma : \{(i \in \mathbb{N} \mid 1 \le i \le n)\} \longrightarrow \{(\sigma(i \in \mathbb{N}) \mid 1 \le i \le n)\}$$

and,

$$t(\sigma) := (\sigma(i \in \mathbb{N}) \mid 1 \le i \le n) \longrightarrow (i \in \mathbb{N} \mid 1 \le i \le n).$$

In particular, skew-symmetry $\Sigma^K_{i\,\sigma(i)} = -\Sigma^K_{\sigma(i)\,i} \,|\, i \geq \sigma(i)$ implies non-vanishing monomials given by:

$$\begin{cases} \displaystyle \sum_{\sigma=1}^{n!/\left(\frac{n}{2}\right)!\,2^{\frac{n}{2}}} (-1)^{t(\sigma)} \prod_{i=1}^{\frac{n}{2}} \Sigma^K_{\tau_1(2i-1)\tau_1(2i)} \prod_{i=1}^{\frac{n}{2}} \Sigma^K_{\tau_2(2i)\tau_2(2i-1)} & \left| \begin{array}{l} \exists\, \Sigma^K_{\tau_1(2i-1)\tau_1(2i)} \\ \quad = -\Sigma^K_{\tau_2(2i)\tau_2(2i-1)}, \\ \forall\, \Sigma^K_{\tau_2(2i)\tau_2(2i-1)}; \\ t(\sigma) = \text{even for } \frac{n}{2} := \text{even}, \\ t(\sigma) = \text{odd for } \frac{n}{2} := \text{odd} \end{array} \right. \\ \quad + \\ \displaystyle \sum_{\sigma=1}^{2\binom{n!/\left(\frac{n}{2}\right)!\,2^{\frac{n}{2}}}{2}} (-1)^{t(\sigma)} \prod_{i=1}^{n} \Sigma^K_{i\,\sigma(i)} & \left| \begin{array}{l} t(\sigma) = \text{odd for } \frac{n}{2} := \text{even}, \\ t(\sigma) = \text{even for } \frac{n}{2} := \text{odd}. \end{array} \right. \end{cases}$$

That is,

$$\begin{cases} \displaystyle\sum_{\sigma=1}^{n!/\left(\frac{n}{2}\right)!\,2^{\frac{n}{2}}} (-1)^{t(\sigma)+\frac{n}{2}} \left(\prod_{i=1}^{\frac{n}{2}} \Sigma^{K}_{\tau_1(2i-1)\tau_1(2i)} \right)^2 & \left| \begin{array}{l} t(\sigma)=\text{even for } \frac{n}{2}:=\text{even}, \\ t(\sigma)=\text{odd for } \frac{n}{2}:=\text{odd} \end{array} \right. \\ + \\ \displaystyle\sum_{\sigma=1}^{2\binom{n!/\left(\frac{n}{2}\right)!\,2^{\frac{n}{2}}}{2}} (-1)^{t(\sigma)} \prod_{i=1}^{n} \Sigma^{K}_{i\,\sigma(i)} & \left| \begin{array}{l} t(\sigma)=\text{odd for } \frac{n}{2}:=\text{even}, \\ t(\sigma)=\text{even for } \frac{n}{2}:=\text{odd}. \end{array} \right. \end{cases}$$

Furthermore, the non-vanishing monomials are given by

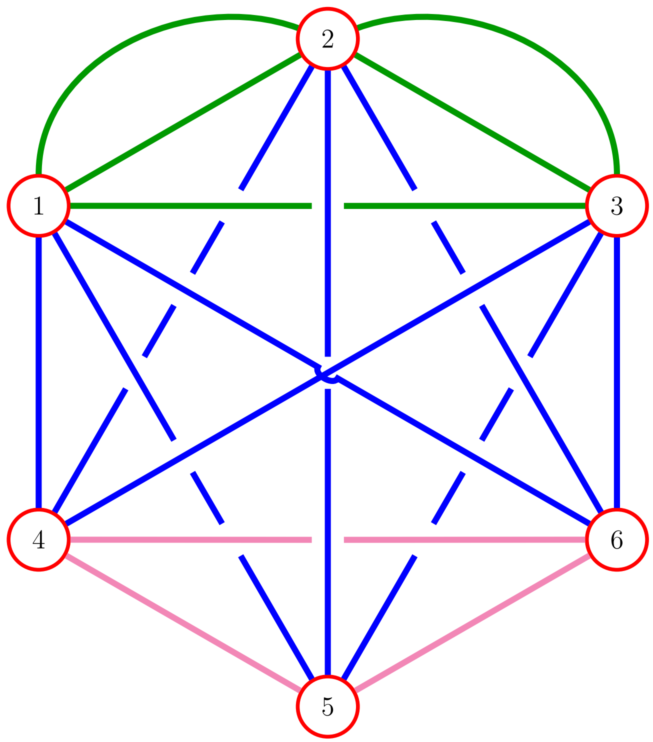
$$\begin{cases} n!\Big/\left(\frac{n}{2}\right)!\,2^{\frac{n}{2}} \\ \qquad \displaystyle\sum_{\widetilde{\sigma}=[\widetilde{\sigma}]=1} (-1)^{t(\sigma)+\frac{n}{2}+2\times t(\widetilde{\sigma})} \left(\prod_{\ell=1}^{\frac{n}{2}} \Sigma^{K}_{\widetilde{\sigma}(2\ell-1)\widetilde{\sigma}(2\ell)} \right)^{2} \Bigg| \begin{array}{l} t(\sigma)=\text{even for } \frac{n}{2}:=\text{even}, \\ t(\sigma)=\text{odd for } \frac{n}{2}:=\text{odd} \end{array} \\ \qquad\qquad\qquad + \\ \qquad \begin{pmatrix} n!\Big/\left(\frac{n}{2}\right)!\,2^{\frac{n}{2}} \\ 2 \end{pmatrix} \\ 2 \times \displaystyle\sum_{\{\widetilde{\sigma}=[\widetilde{\sigma}]\neq\widetilde{\tau}=[\widetilde{\tau}]\}=1} (-1)^{t(\widetilde{\sigma})} \prod_{\ell=1}^{\frac{n}{2}} \Sigma^{K}_{\widetilde{\sigma}(2\ell-1)\widetilde{\sigma}(2\ell)} \times (-1)^{t(\widetilde{\tau})} \prod_{\ell=1}^{\frac{n}{2}} \Sigma^{K}_{\widetilde{\tau}(2\ell-1)\widetilde{\tau}(2\ell)} \\ \forall\,[\sigma]:=\mathcal{S}_n\Big/\left(\mathcal{S}_{\frac{n}{2}}\times\mathcal{S}_2^{\frac{n}{2}}\right). \end{cases}$$

That is,

$$\begin{cases} n!\Big/\left(\dfrac{n}{2}\right)!\,2^{\frac{n}{2}} \\ \quad \displaystyle\sum_{\widetilde{\sigma}=[\widetilde{\sigma}]=1} (-1)^{2\times t(\widetilde{\sigma})} \left(\prod_{\ell=1}^{\frac{n}{2}} \Sigma^{K}_{\widetilde{\sigma}(2\ell-1)\widetilde{\sigma}(2\ell)} \right)^{2} \quad \Bigg|\, \begin{array}{l} t(\widetilde{\sigma}) := (\widetilde{\sigma}(i \in \mathbb{N}) \,|\, 1 \le i \le n) \\ \qquad\qquad \longrightarrow (i \in \mathbb{N} \,|\, 1 \le i \le n) \end{array} \\ \qquad\qquad\qquad + \\ \quad \begin{pmatrix} n!\Big/\left(\dfrac{n}{2}\right)!\,2^{\frac{n}{2}} \\ 2 \end{pmatrix} \\ 2 \;\times \displaystyle\sum_{\{\widetilde{\sigma}=[\widetilde{\sigma}]\neq\widetilde{\tau}=[\widetilde{\tau}]\}=1} (-1)^{t(\widetilde{\sigma})+t(\widetilde{\tau})} \prod_{\ell=1}^{\frac{n}{2}} \Sigma^{K}_{\widetilde{\sigma}(2\ell-1)\widetilde{\sigma}(2\ell)} \prod_{\ell=1}^{\frac{n}{2}} \Sigma^{K}_{\widetilde{\tau}(2\ell-1)\widetilde{\tau}(2\ell)} \\ \forall\,[\sigma] := \mathcal{S}_n \Big/ \left(\mathcal{S}_{\frac{n}{2}} \times \mathcal{S}_2^{\frac{n}{2}}\right) \end{cases}$$

$$= \left(\sum_{\sigma=[\sigma]} (-1)^{t(\sigma)} \prod_{\ell=1}^{\frac{n}{2}} \Sigma^{K}_{\sigma(2\ell-1)\sigma(2\ell)} \right)^{2} \quad \Bigg|\, \begin{array}{l} t(\sigma) := (\sigma(i \in \mathbb{N}) \,|\, 1 \le i \le n) \\ \qquad\qquad \longrightarrow (i \in \mathbb{N} \,|\, 1 \le i \le n) \end{array}$$

$$\forall\,[\sigma] := \mathcal{S}_n \Big/ \left(\mathcal{S}_{\frac{n}{2}} \times \mathcal{S}_2^{\frac{n}{2}}\right). \qquad \square$$



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

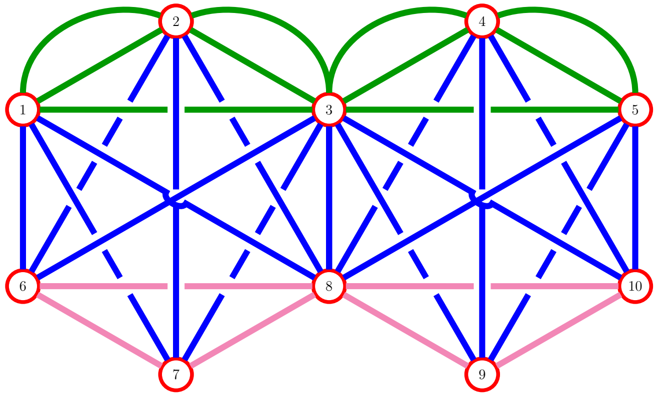
$0 :=$ non-adjoined i, j

$1, 1, 1 :=$ adjoined i, j.

To see $|\mathrm{Pf}(\Sigma^K)| = Z$, write on the non-vanishings, by partitions $\sigma = [\sigma]$, $\forall\, D$:

$$Z = \sum_{\sigma = [\sigma]} \sum_{D(\sigma)} \varepsilon^K_{\sigma(1)\sigma(2)}\, \omega_{\sigma(1)\sigma(2)} \cdots \varepsilon^K_{\sigma(n-1)\sigma(n)}\, \omega_{\sigma(n-1)\sigma(n)} =$$

$$= \sum_{\sigma = [\sigma]} \sum_{D(\sigma)} \prod_{\ell=1}^{\frac{n}{2}} \varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)}\, \omega_{\sigma(2\ell-1)\sigma(2\ell)}.$$



0 := non-adjoined i, j

$1, 1, 1$:= adjoined i, j

$$\left(\begin{array}{cccccc|cccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

But the derived Pfaffian is given by

$$\mathsf{Pf}\big(\Sigma^K\big) = \begin{cases} \displaystyle\sum_{\substack{\sigma\,\in\,\mathcal{S}_n/(\mathcal{S}_{\frac{n}{2}}\times\mathcal{S}_2^{\frac{n}{2}}) \\ \| \\ [\sigma]}} \underbrace{(-1)^{t(\sigma)}\prod_{\ell=1}^{\frac{n}{2}}\Sigma^K_{\sigma(2\ell-1)\sigma(2\ell)}}_{\text{depends only on } \sigma} \end{cases}$$

$$= \sum_{\sigma=[\sigma]} \sum_{D(\sigma)} \overbrace{(-1)^{t(\sigma)}\prod_{\ell=1}^{\frac{n}{2}}\varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)}}^{\text{depends only on } \sigma} \prod_{\ell=1}^{\frac{n}{2}}\omega_{\sigma(2\ell-1)\sigma(2\ell)}.$$

Therefore, write

$$\mathsf{Pf}(\Sigma^K) = \sum_{D(\sigma)} (-1)^{t(\sigma)} \prod_{\ell=1}^{\frac{n}{2}}\varepsilon^K_{\sigma(2\ell-1)\sigma(2\ell)} \prod_{\ell=1}^{\frac{n}{2}}\omega_{\sigma(2\ell-1)\sigma(2\ell)} = \underset{\substack{\| \\ (-1)^{t(\sigma)}}}{(\mathsf{sgn})} \cdot Z\,.$$

where the first equality is well-defined by all partitions $[\sigma]$, $\forall\, \ell \in \mathscr{D}$. □

Corollary.

$$\left\langle \prod_{i,\,j=1}^{k} \boldsymbol{\sigma}(i)\boldsymbol{\sigma}(j) \right\rangle = \mathsf{Pf}\big((\Sigma^K)^{-1}_{\xi\eta}\big) \quad \Bigg| \quad \begin{array}{l} \xi = 1, \ldots, k \\ \eta = 1, \ldots, k. \end{array}$$

Proof. ♡.

Remark. $\mathsf{Pf}(\Sigma^K)$:= partition function, $(\Sigma^K)^{-1}$:= correlation functions.

Remark. Kasteleyn theorem allows polynomial-time; skew-symmetric Gauss elimination algorithm, by $\mathsf{Pf}(BAB^T) = \det(B)\,\mathsf{Pf}(A)$, does $\mathcal{O}(m^3)$ time for Pfaffian of size $2m$ matrix.

1.5 Grassmann integral

Definition. Let $(v_1, \dots, v_n) \in V :=$ vector space, where $\bigwedge^n V :=$ exterior algebra of V is defined by:

(i) Elements:

$$\bigwedge_{i=1}^{n} v_i := v_1 \wedge \cdots \wedge v_n =$$
$$= \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{t(\sigma)} \bigotimes_{i=1}^{n} v_{\sigma(i)} := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{t(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)};$$

(ii) Multiplication:

$$\Big(\bigwedge_{i=1}^{n} v_i\Big) \bigwedge \Big(\bigwedge_{i=1}^{n} w_i\Big) := (v_1 \wedge \cdots \wedge v_n) \wedge (w_1 \wedge \cdots \wedge w_n) =$$
$$= v_1 \wedge \cdots \wedge v_n \wedge w_1 \wedge \cdots \wedge w_n.$$

Then, with chosen *basis* $a = (a_1, \dots, a_n) \in V$, the exterior algebra

$$\textstyle\bigwedge^n V := \{a_i \mid a_i a_j + a_j a_i = 0\}$$

is a Grassmann algebra.

Proposition. The Grassmann algebra $\bigwedge^n V$ is generated by V.

Proof. $\heartsuit$.

Definition. The Grassmann integral on $\bigwedge^n V$ with respect to x is given by

$$\int_{\bigwedge^n V} f = f_x \,, \quad f = f_x\, x + \underbrace{\cdots}_{\text{lower order terms}}$$

where $x \in \bigwedge^n V \subset \mathbb{R}$ is an orientation.

Lemma. $x = \prod_{i=1}^n a_i$ *if* $(a_1, \ldots, a_n)$ *is a basis in* V.

Proposition. *The formal Grassmann constraints are given by the integral*:

$$\int \prod_{i=1}^n a_i \prod_{i=1}^n da_i = (-1)^{\frac{n(n-1)}{2}} \int \prod_{i=1}^n a_i\, da_i = (-1)^{\frac{n(n-1)}{2}}.$$

where:

(i). $$\int \prod_{\nu=1}^n a_{i_\nu}\, da = \begin{cases} 0 & , \quad k < n \\ (-1)^{t(\sigma)} & , \quad k = n \end{cases}$$
$$\sigma : (i_1, \ldots, i_n) \longrightarrow (1, \ldots, n).$$

(ii). $$da = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^n da_i\,.$$

Proof. $\heartsuit$.

Theorem. *Let* $a = (a_i) \in V$ *be basis, where* $A^* = A_a = \exp\left(\frac{1}{2}\sum_{ij} a_i A_{ij} a_j\right)$ *satisfies the Grassmann constraints, uniquely generalizing* A^* *for all Kasteleyn matrices* A *satisfying the Grassmann constraints. Then:*

(i) $$Pf(A) = \int_{\bigwedge^n V} \exp\left(\frac{1}{2}\sum_{ij} a_i A_{ij} a_j\right) da.$$

(ii) $$Pf\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} = \det(A).$$

(iii) $$(Pf(A))^2 = \det(A).$$

(iv) $$\frac{\partial}{\partial A_{i_1 j_1}} \cdots \frac{\partial}{\partial A_{i_k j_k}} Pf(A) =$$
$$= Pf(A) \cdot Pf\left((\Sigma^K)^{-1}_{ab}\right) \Big|\, \begin{matrix} a = i_1, \ldots, i_k \\ b = j_1, \ldots, j_k\,. \end{matrix}$$

Proof.

(i). Write:

$$\int\limits_{\bigwedge^n V} \exp\left(\frac{1}{2}\langle a, Aa\rangle\right) da \;=\; \frac{1}{(\frac{n}{2})!}\,\frac{1}{2^{\frac{n}{2}}} \int\limits_{\bigwedge^n V} \langle a, Aa\rangle^{\frac{n}{2}}\, da$$

such that

$$\begin{aligned} \int \langle a, Aa\rangle^{\frac{n}{2}}\, da \quad &= \int a_{i_1} a_{j_1} \cdots a_{i_{\frac{n}{2}}} a_{j_{\frac{n}{2}}}\, A_{i_1\, j_1} \cdots A_{i_{\frac{n}{2}}\, j_{\frac{n}{2}}} \\ &= (-1)^{\sigma} A_{i_1\, j_1} \cdots A_{i_{\frac{n}{2}} j_{\frac{n}{2}}} \\ &\quad \sigma : \left((i_1, j_1), \ldots, \left(i_{\frac{n}{2}}, j_{\frac{n}{2}}\right)\right) \to (1, \ldots, n)\,. \end{aligned}$$

This implies

$$\int\limits_{\bigwedge^n V} \exp\left(\frac{1}{2}\langle a, Aa\rangle\right) da \;=\; \frac{1}{(\frac{n}{2})!}\,\frac{1}{2^{\frac{n}{2}}}\,\mathsf{Pf}(A)\,.$$

Use the integral formula to prove II, III, IV. ♡.

Proof (hints).

(ii). Choosing splitting $V = W \oplus W^*$ by matrix block structure, such that the Grassmann algebra of $V \cong$ algebra (tensor product) generated by $c_i, b_i \,|\, i=1,\ldots,\frac{n}{2}$, with relations $c_i c_j = -c_j c_i$, $c_i b_j = -b_j c_i$, and $b_1 b_j = -b_j b_i$:

$$(a_1, \ldots, a_n) =$$
$$= \big(\underbrace{c_1, \ldots, c_{\frac{n}{2}}}_{\text{basis in } W}, \underbrace{b_1, \ldots, b_{\frac{n}{2}}}_{\text{basis in } W^*}\big)$$

then

$$\left\langle a\,,\ \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} a \right\rangle = 2\,\langle c, Ab\rangle\,.$$

Hence, prove

$$\int_{\bigwedge^n (W \oplus W^*)} \exp\left(\langle c, Ab\rangle\right) dc\, db = \det\left(A\right).$$

(iii). Similar.

(iv). $\displaystyle\int \exp\left(\frac{1}{2}\langle a, Aa\rangle + \langle a, \eta\rangle\right) da \;=$

$$= \int \exp\left(\frac{1}{2}\left\langle a + A^{-1}\eta,\ A\left(a + A^{-1}\eta\right)\right\rangle - \frac{1}{2}\left\langle \eta,\ A^{-1}\eta\right\rangle\right) da$$

$$= \mathsf{Pf}(A)\ \exp\left(-\frac{1}{2}\left\langle \eta,\ A^{-1}\eta\right\rangle\right).$$

$$\frac{\partial}{\partial A_{i_1 j_1}} \cdots \frac{\partial}{\partial A_{i_k j_k}}\ \mathsf{Pf}(A) \;=$$

$$= \int a_{i_1} a_{j_1} \cdots a_{i_k} a_{j_k} \exp\left(\frac{1}{2}\langle a, Aa\rangle\right) da$$

$$= \left(\frac{\partial}{\partial \eta}\right)^{2k} \int \exp\left(\frac{1}{2}\langle a, Aa\rangle + \langle \eta, a\rangle\right) da.$$

Prove the Pfaffian formula for the correlation functions. $\heartsuit$.

1.6 Kasteleyn solution for bipartite graph

Given $\Gamma \subset \mathbb{R}^2$ endowed with configuration and Kasteleyn orientation

$$Z_\Gamma = \varepsilon_\Gamma^K \int \exp\Big(\frac{1}{2}\sum_{ij} a_i \,(\Sigma_\Gamma^K)_{ij}\, a_j\Big) da \;\Bigg|\; \begin{array}{l} \varepsilon_\Gamma^K := (-1)^\sigma\, \varepsilon^K_{\sigma_1\sigma_2} \cdots \varepsilon^K_{\sigma_{n-1}\sigma_n} \in \{\pm 1\} \\ n = |V(\Gamma)|. \end{array}$$

In addition, $\Gamma :=$ bipartite implies

$$\Sigma_\Gamma^K = \begin{pmatrix} 0 & B_\Gamma^K \\ -(B^K)_\Gamma^t & 0 \end{pmatrix} \;\Bigg|\; \begin{array}{l} B^K : \mathbb{R}^{V_\circ(\Gamma)} \to \mathbb{R}^{V_\bullet(\Gamma)} \\ \mathbb{R}^{V(\Gamma)} = \mathbb{R}^{V_\bullet(\Gamma)} \oplus \mathbb{R}^{V_\circ(\Gamma)} \\ \dim(\mathbb{R}^{V_\bullet(\Gamma)}) = \dim(\mathbb{R}^{V_\circ(\Gamma)}) = \frac{n}{2} \\ V(\Gamma) = V_\bullet(\Gamma) \sqcup V_\circ(\Gamma),\ |V(\Gamma)| = n. \end{array}$$

Identifying $V_\bullet(\Gamma)$, $V_\circ(\Gamma)$ via a diagram $\{b\} \sim \{\omega\}$ with "hole"

$$\Sigma_\Gamma^K = \begin{pmatrix} 0 & C_\Gamma^K \\ -(C_\Gamma^K)^t & 0 \end{pmatrix} \text{ for } \mathbb{R}^{V_\bullet(\Gamma)} \oplus \mathbb{R}^{V_\circ(\Gamma)} \hookleftarrow \;\Big|\; C_\Gamma^K := \mathbb{R}^{V_\circ(\Gamma)} \hookleftarrow$$

where $\hookleftarrow$ denotes recursive invocations.

Then:

(i). $Z_\Gamma = |\det(C_\Gamma^K)|$

(ii). $\left\langle \boldsymbol{\sigma}_{b_1 w_1} \cdots \boldsymbol{\sigma}_{b_k w_k} \right\rangle = \dfrac{\partial}{\partial\, \boldsymbol{\omega}(b_1 w_1)} \cdots \dfrac{\partial}{\partial\, \boldsymbol{\omega}(b_k w_k)} \ln Z_\Gamma =$

$$= \det\Big(\big((C_\Gamma^K)^{-1}\big)_{\hat{b}\, w}\Big) \;\; \Bigg| \begin{array}{l} \hat{b} = \hat{b}_1, \ldots, \hat{b}_k \\ w = w_1, \ldots, w_k \end{array}$$

where the inverse-matrix element $\hat{b}$ is the white vertex identified with b.

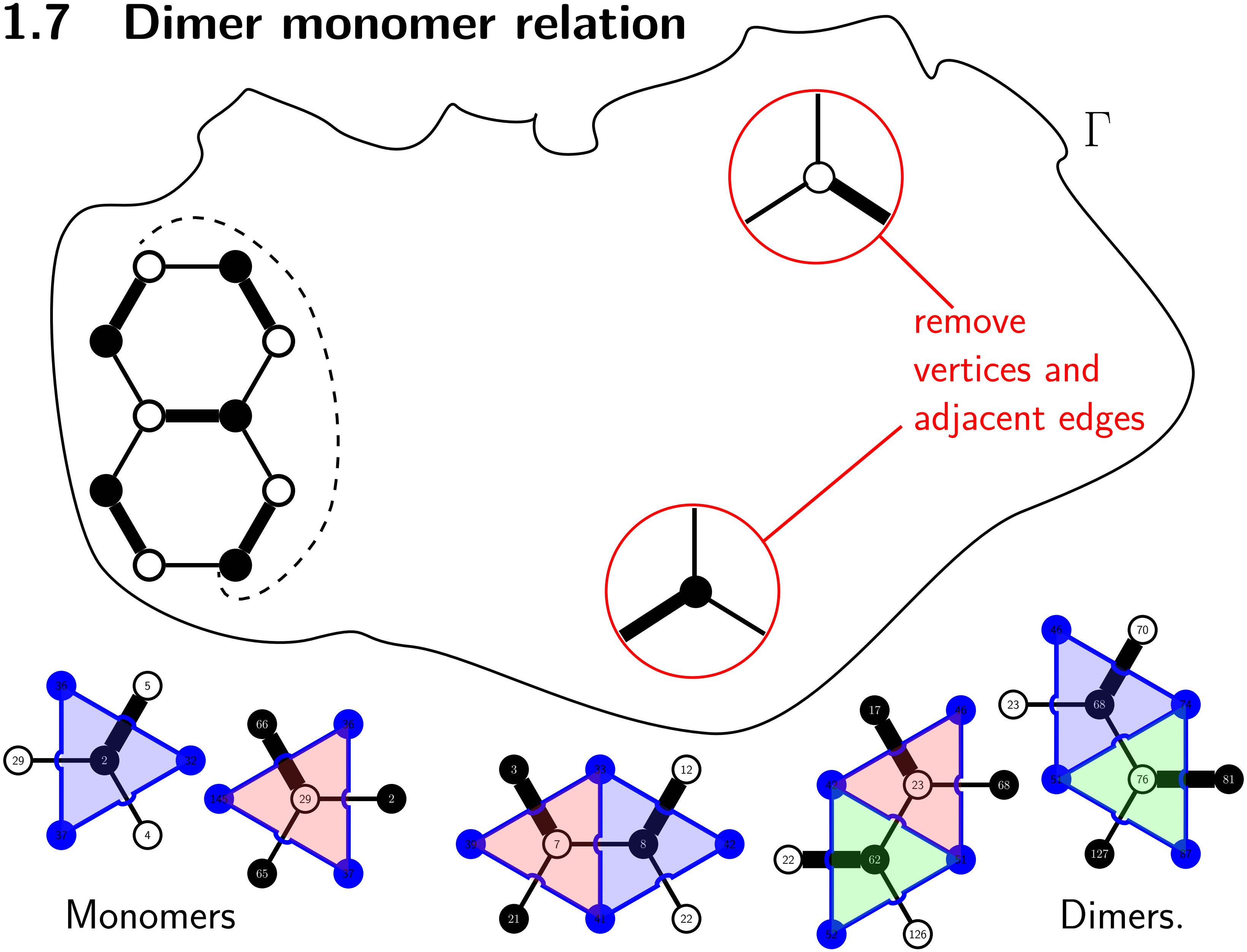
Remark. The "Physical" meaning of (ii) is given by

$$\left\langle \boldsymbol{\sigma}_{b_1 w_1} \cdots \boldsymbol{\sigma}_{b_k w_k} \right\rangle =$$

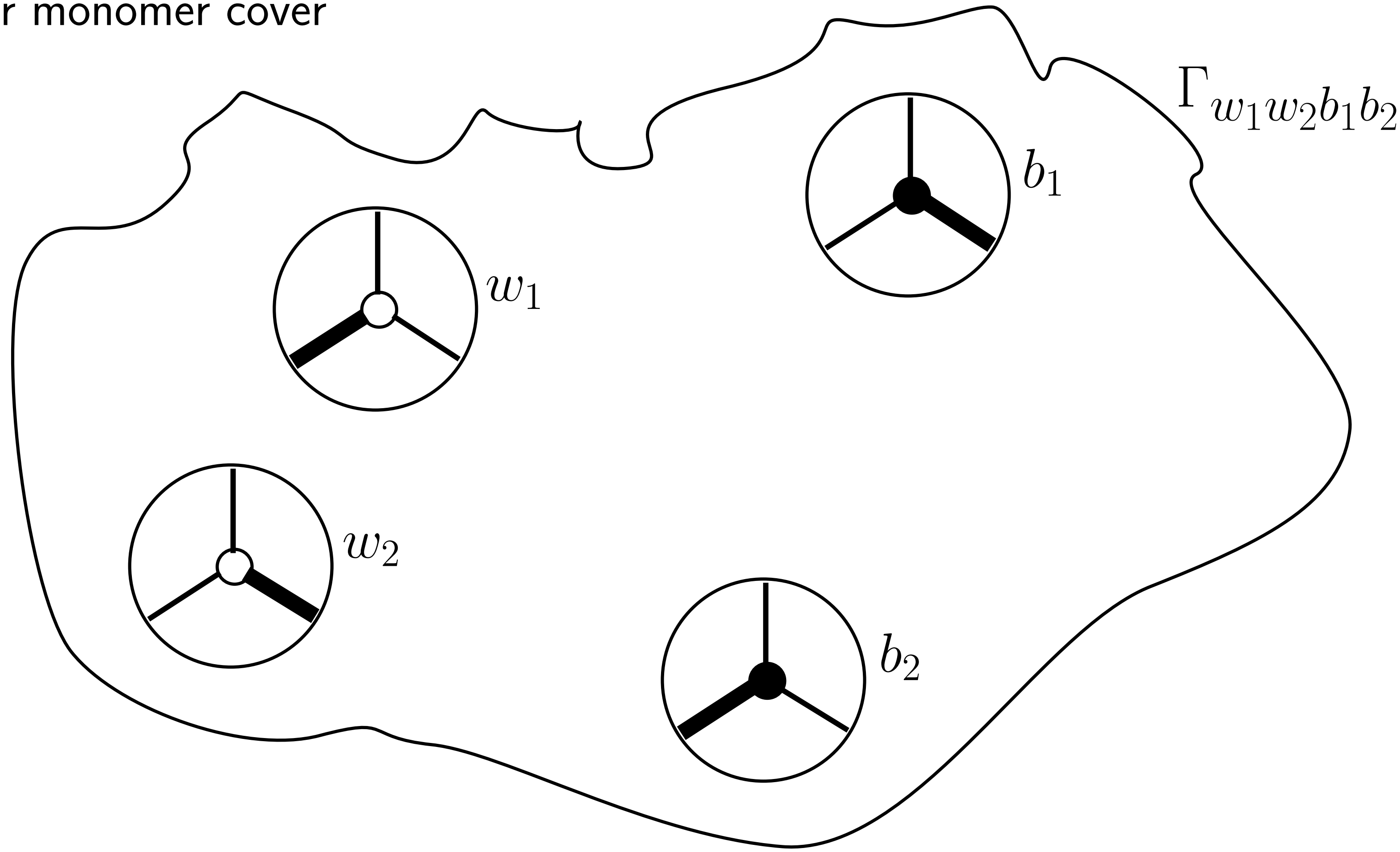
$$= \int \boldsymbol{\psi}^*_{b_1 w_1} \cdots \boldsymbol{\psi}^*_{b_k w_k} \times \exp\big(\boldsymbol{\psi}^* C_\Gamma^K \boldsymbol{\psi}\big) \times \left(\int \exp\big(\boldsymbol{\psi}^* C_\Gamma^K \boldsymbol{\psi}\big)\, d\boldsymbol{\psi}^*\, d\boldsymbol{\psi} \right).$$

That is, it corresponds to correlation functions in free fermionic model.

1.7 Dimer monomer relation



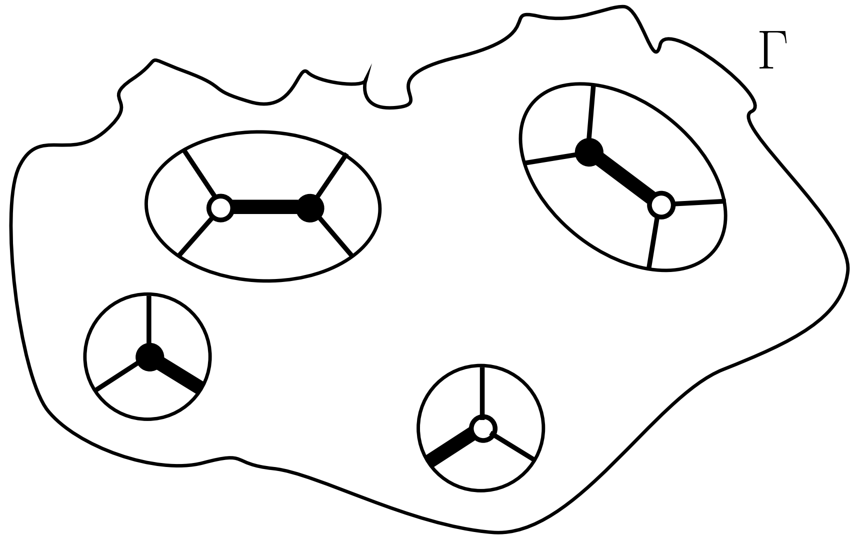
For monomer cover



the monomer-monomer correlation functions are given by

$$M_{b_1 \cdots b_n w_1 \cdots w_n} = \frac{Z_{\Gamma_{w_1 \cdots w_n b_1 \cdots b_n}}}{Z_\Gamma} .$$

Remark. If w_1 and b_1 are adjacent, then we have a dimer



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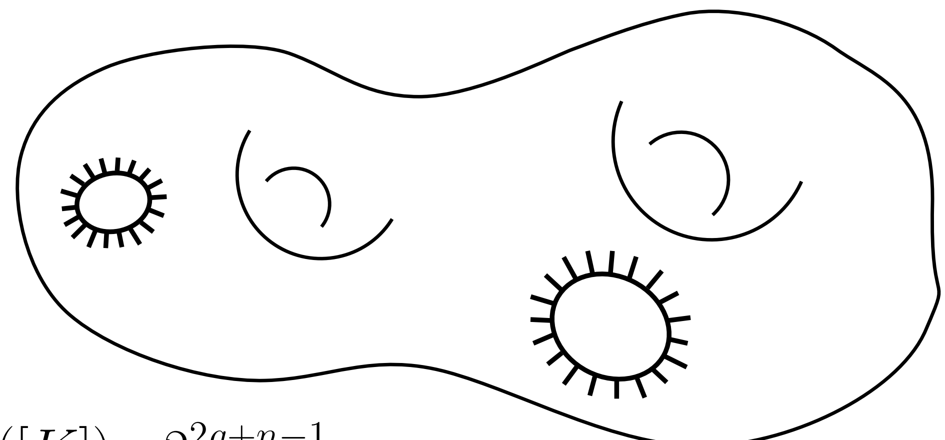
Remark. The monomer-monomer correlation functions are a special case of dimer models on surfaces with nontrivial fundamental group.



In this case, $\#([K])=2^{2g+n-1}$.

.

1.8 Partition function as sum of Pfaffians

Theorem.

$$Z^{dimers}_{\Gamma\subset\overline{\mathcal{M}}_g} = \frac{1}{2^g}\sum_{[K]} \mathsf{Arf}(q^K_{D_0})\varepsilon^K(D_0)\cdot \mathsf{Pf}(\Sigma^K) \quad \Big| \, \mathsf{Arf}(\cdot) := \mathsf{sgn}.$$

where:

$[K]$:= equivalence classes of Kasteleyn orientations, 2^{2g} in total

$[q^K_{D_0}]$:= quadratic form on $\mathcal{H}^1(\Sigma,\mathbb{Z}_2)$ associated with Kasteleyn orientation on reference configuration D_0

$$\boldsymbol{\varepsilon}^K(D_0) := (-1)^{\boldsymbol{\sigma}}\boldsymbol{\varepsilon}^K_{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}\cdots\boldsymbol{\varepsilon}^K_{\boldsymbol{\sigma}_{n-1}\boldsymbol{\sigma}_n}$$

$$\boldsymbol{\sigma}\in S_n, \;\; [\boldsymbol{\sigma}] := S_n\big/\big(S_2^{\frac{n}{2}}\times S_{\frac{n}{2}}\big)$$

$n :=$ number of vertices.

Proof. ♡.

Remark. In particular:

I. For bipartite graphs on $\overline{\mathcal{M}}_g$:

height function :=

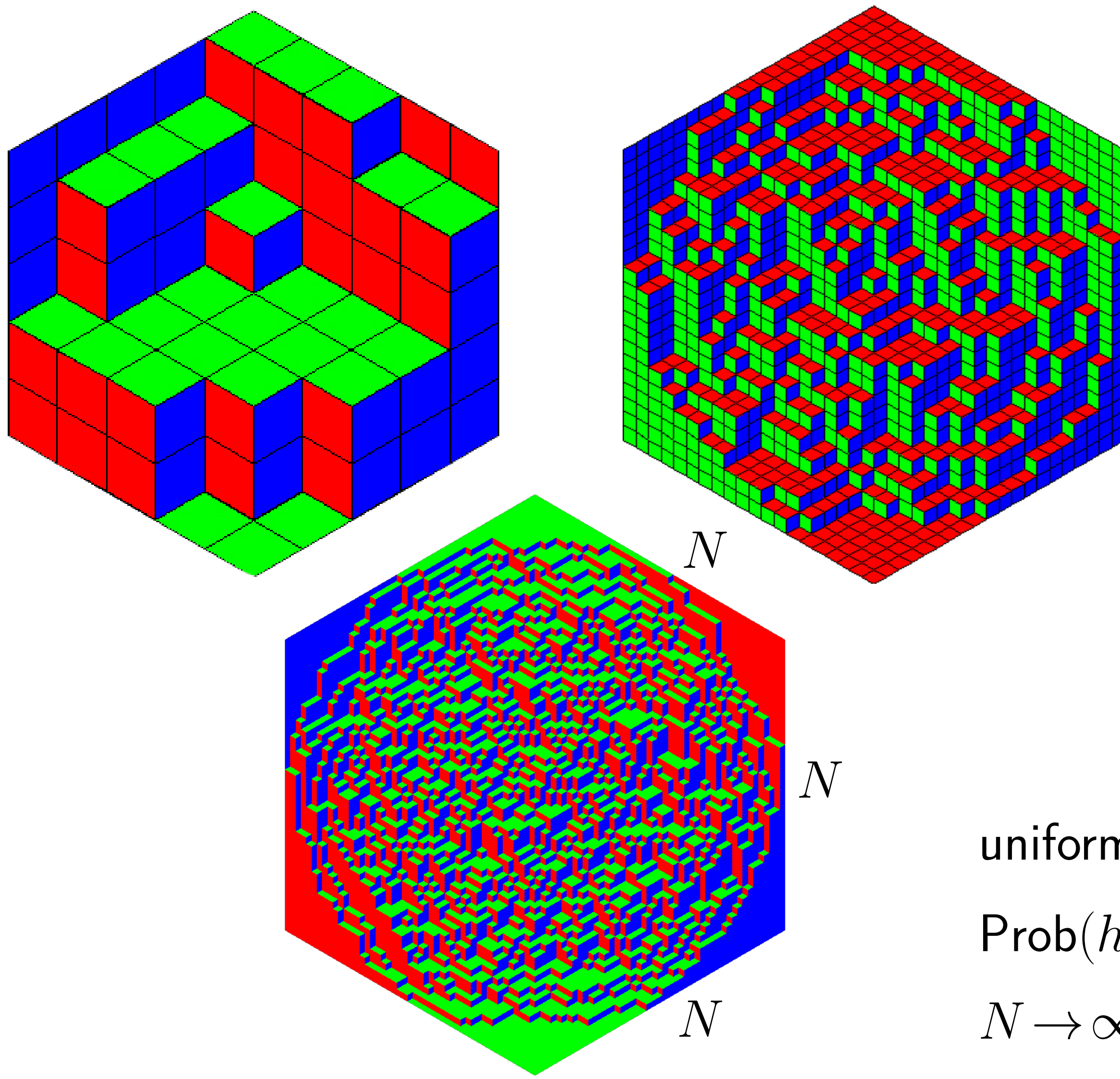
$=$ section of the non-trivial $\mathbb{Z}$-bundle.

II. For fundamental cycles $(a_1, \ldots, a_g, b_1, \ldots, b_g)$:

$$Z\big(\mathcal{H}_{a_1}, \ldots, \mathcal{H}_{a_g}, \mathcal{H}_{b_1}, \ldots, \mathcal{H}_{b_g}\big) = \\ = \sum_D \prod_{\ell \in D} \boldsymbol{\omega}(\ell) \prod_{i=1}^{g} \exp\Big(\sum_i \mathcal{H}_{a_i} \Delta_{a_i} h + \\ + \sum_i \mathcal{H}_{b_i} \Delta_{b_i} h \Big)$$

where $\Delta_C h :=$ change in height function along every noncontractible cycle C on $\overline{\mathcal{M}}_g$.

1.9 Thermodynamic limit of bulk interactions

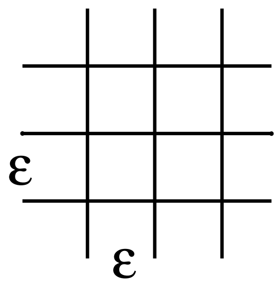


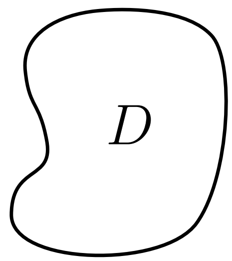
uniform measure

$\text{Prob}(h) = \frac{1}{|\mathcal{H}_\Gamma|}$

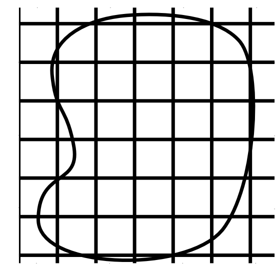
$N \to \infty$.

Theorem (Schur process; Okounkov & R). Let $\varphi_\varepsilon : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2 \mid D \subset \mathbb{R}^2$.





such that: $\begin{cases} \varepsilon \to 0, \ |D_\varepsilon| \to \infty \\ D_\varepsilon := \varphi_\varepsilon\left(\mathbb{Z}^2\right) \cap D \end{cases}$



.

Then, for stacks of cubes with measure

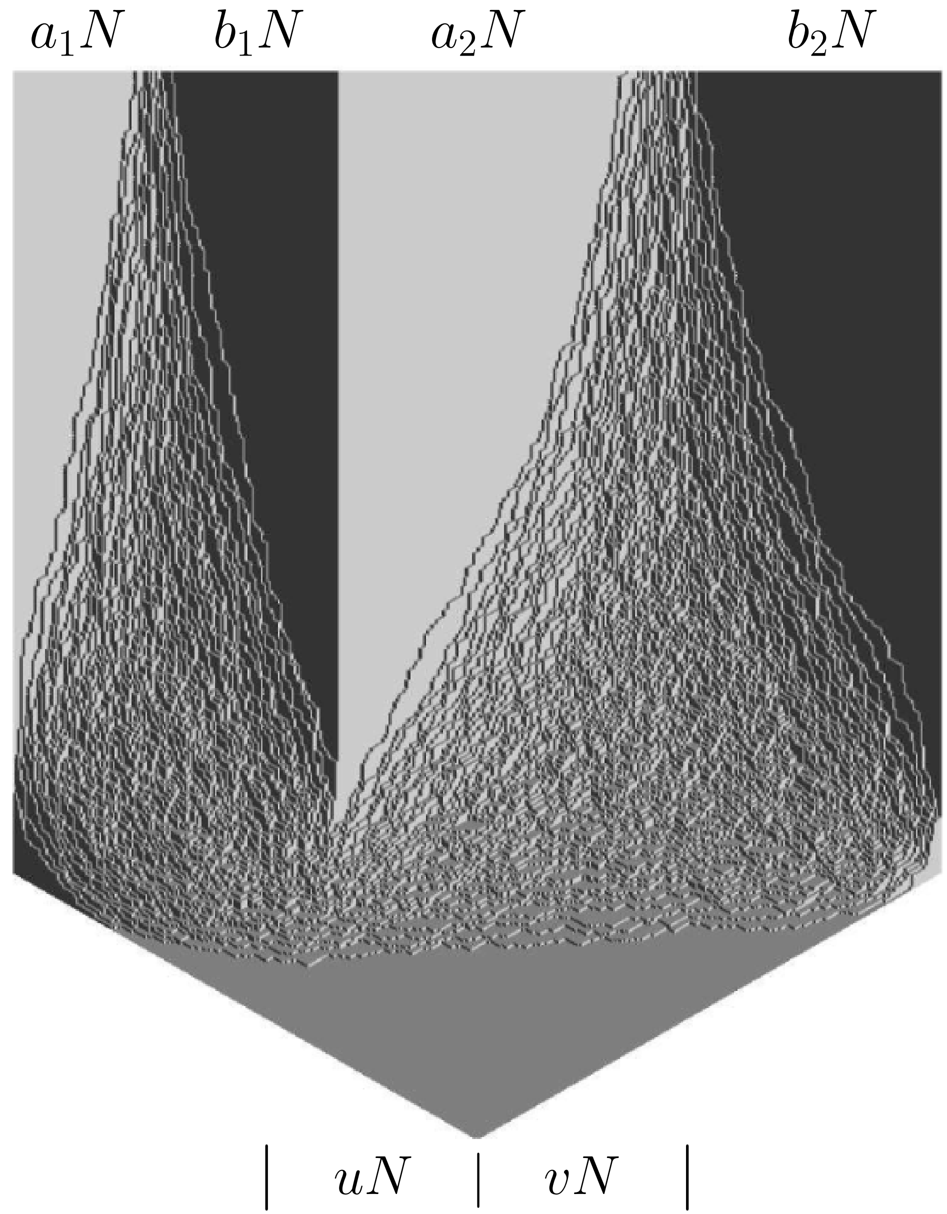
$$\textit{Prob}(\pi) = \frac{\prod_t q_t^{\pi(t)}}{\sum_\pi \prod_t q_t^{\pi(t)}} \quad \Bigg| \ \pi \in \mathcal{H}_\Gamma \, , \ \pi \cong D,$$

there is an existence of

$$\textit{Thermodynamic limit } (|D_\varepsilon| \to \infty) \, +$$

$$+ \ \textit{Scaling limit } (q = e^{-\varepsilon}, \ \varepsilon \to +0).$$

Proof. $\heartsuit$.



where $u+v = a_1+a_2+b_1+b_2,\ N = \varepsilon^{-1},\ q = e^{-\varepsilon}$.

1.10 Enumeration of perfect matchings in graphs

Involves computation of matching polynomials using the derived Kasteleyn determinantal method:

- $\Gamma \subset \overline{\mathcal{M}}_g := m \times n$ planar square lattice without closed boundary:

$$\mu(\Gamma; m, n) :=$$

$$2^{\frac{mn}{2}} \prod_{i=1}^{m} \prod_{j=1}^{\frac{n}{2}} \sqrt{\cos^2\left(\frac{\pi i}{m+1}\right) + \cos^2\left(\frac{\pi j}{n+1}\right)}$$

$n :=$ even.

Or,

$= 0$, for $n :=$ odd.

Proof. $\heartsuit$.

- $\Gamma \subset \overline{\mathcal{M}}_g := m \times n$ cylindrical square lattice:

$$\mu(\Gamma; m, n) :=$$

$$2^{\frac{mn}{2}} \prod_{i=1}^{m} \prod_{j=1}^{\frac{n}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{m}\right) + \cos^2\left(\frac{\pi j}{n+1}\right)}$$

$n :=$ even.

Or,

$$\mu(\Gamma; m, n) :=$$

$$2^{\frac{mn}{2} - \frac{m}{2} + 1} \prod_{i=1}^{m} \prod_{j=1}^{\frac{n}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{m}\right) + \cos^2\left(\frac{\pi j}{n+1}\right)}$$

$n :=$ odd.

Proof. $\heartsuit$.

- $\Gamma \subset \overline{\mathcal{M}}_g := m \times n$ toroidal square lattice:

$$\boldsymbol{\mu}(\Gamma; m, n) :=$$

$$2^{\frac{mn}{2}-1} \left(\begin{array}{c} \prod_{i=1}^{m} \prod_{j=1}^{\frac{n}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{m}\right) + \sin^2\left(\frac{2\pi j}{n}\right)} \\ + \\ \prod_{i=1}^{m} \prod_{j=1}^{\frac{n}{2}} \sqrt{\sin^2\left(\frac{2\pi i}{m}\right) + \sin^2\left(\frac{\pi(2j-1)}{n}\right)} \\ + \\ \prod_{i=1}^{m} \prod_{j=1}^{\frac{n}{2}} \sqrt{\sin^2\left(\frac{\pi(2i-1)}{m}\right) + \sin^2\left(\frac{\pi(2j-1)}{n}\right)} \end{array} \right)$$

$n :=$ even.

Or, $= 0$, for $n :=$ odd.

Proof. $\heartsuit$.

- $\Gamma \subset \overline{\mathcal{M}}_g := 6 \times 8$ planar square lattice without closed boundary:

$$
\boldsymbol{\mu}(\Gamma; m, n) :=
16777216 \left(\frac{1}{4} + \sin^2\left(\frac{\pi}{14}\right)\right) \left(\sin^2\left(\frac{\pi}{18}\right) + \sin^2\left(\frac{\pi}{14}\right)\right) \left(\frac{1}{4} + \sin^2\left(\frac{3\pi}{14}\right)\right)
\times \left(\sin^2\left(\frac{\pi}{18}\right) + \sin^2\left(\frac{3\pi}{14}\right)\right) \left(\frac{1}{4} + \cos^2\left(\frac{\pi}{7}\right)\right) \left(\cos^2\left(\frac{\pi}{9}\right) + \cos^2\left(\frac{\pi}{7}\right)\right)
\times \left(\cos^2\left(\frac{\pi}{7}\right) + \cos^2\left(\frac{2\pi}{9}\right)\right) \left(\sin^2\left(\frac{\pi}{18}\right) + \cos^2\left(\frac{\pi}{7}\right)\right)
\times \left(\sin^2\left(\frac{\pi}{14}\right) + \cos^2\left(\frac{\pi}{9}\right)\right) \left(\sin^2\left(\frac{\pi}{14}\right) + \cos^2\left(\frac{2\pi}{9}\right)\right)
\times \left(\sin^2\left(\frac{3\pi}{14}\right) + \cos^2\left(\frac{\pi}{9}\right)\right) \left(\sin^2\left(\frac{3\pi}{14}\right) + \cos^2\left(\frac{2\pi}{9}\right)\right)
$$

- $\Gamma \subset \overline{\mathcal{M}}_g := 6 \times 8$ cylindrical square lattice:

$$\mu(\Gamma; m, n) :=$$
$$5242880 \left(\frac{1}{4} + \sin^2\left(\frac{\pi}{18}\right)\right)^2 \left(1 + \sin^2\left(\frac{\pi}{18}\right)\right) \left(\frac{1}{4} + \cos^2\left(\frac{\pi}{9}\right)\right)^2$$
$$\left(1 + \cos^2\left(\frac{\pi}{9}\right)\right) \left(\frac{1}{4} + \cos^2\left(\frac{2\pi}{9}\right)\right)^2 \left(1 + \cos^2\left(\frac{2\pi}{9}\right)\right)$$

- $\Gamma \subset \overline{\mathcal{M}}_g := 6 \times 8$ toroidal square lattice:

$$\mu(\Gamma; m, n) :=$$
$$8388608\left(\frac{18225}{131072} + \left(\frac{1}{4} + \sin^2\left(\frac{\pi}{8}\right)\right)^4 \left(1 + \sin^2\left(\frac{\pi}{8}\right)\right)^2 \left(\frac{1}{4} + \cos^2\left(\frac{\pi}{8}\right)\right)^4\right.$$
$$\left.\left(1 + \cos^2\left(\frac{\pi}{8}\right)\right)^2 + \sin^4\left(\frac{\pi}{8}\right)\left(\frac{3}{4} + \sin^2\left(\frac{\pi}{8}\right)\right)^4 \cos^4\left(\frac{\pi}{8}\right)\left(\frac{3}{4} + \cos^2\left(\frac{\pi}{8}\right)\right)^4\right)$$

2 Special cases

Points:

(i). Reformulate Kasteleyn Grassmann integral by transfer-matrices (for special domains)

(ii). Compute inverse of Kasteleyn operator

(iii). Find scaling limit

(iv). Derive variational principle for limit topologies.

2.1 Grassmann integral kernel

Let $\bigwedge^n V := \{a_i \mid a_i a_j + a_j a_i = 0\}$ be Grassmann algebra generated by V, for a chosen basis $a = (a_1, \ldots, a_n) \in V$.

For vectors $\psi \in \bigwedge^n V$, write

$$\psi(a) = \sum_{k=0}^{n} \sum_{i_1 < \cdots < i_k}^{n} a_{i_1} \cdots a_{i_k} \, \psi_{\{i\}_<}$$

$$\{i\}_< := \{i_1, \ldots, i_k\}, \quad i_1 < \cdots < i_k \, .$$

For vectors $\varphi \in \bigwedge^n V^*$, where a^* is basis of dual vector space V^*, write

$$\varphi\big(a^*\big) = \sum_{k=0}^{n} \sum_{\{i\}_>}^{n} \varphi_{\{i\}_>} \, a^*_{\{i\}_>} \, .$$

Pair $\bigwedge^n V^* \otimes \bigwedge^n V \to \mathbb{R}$ by

$$\big\langle \varphi\big(a^*\big), \, \psi(a) \big\rangle \stackrel{\text{def}}{=\!=} \sum_{\{i\}_<}^{n} \varphi_{i_k \cdots i_1} \, \psi_{i_1 \cdots i_k} \, .$$

Choosing

$$a_1, \ldots, a_n \in \bigwedge^n V, \quad a_n^*, \ldots, a_1^* \in \bigwedge^n V^*$$

and

$$a_n^*, \ldots, a_1^*,\, a_1, \ldots, a_n \in \bigwedge^n V^* \otimes \bigwedge^n V$$

then

$$\int \prod_{\nu=1}^{n} a_{i_\nu}^* \; \prod_{\nu=1}^{n} a_{j_\nu} \; da^* \, da \; = \begin{cases} 0 & , \quad k \neq n \\ (-1)^{\left(\sigma + \tau + \frac{n(n-1)}{2}\right)} & , \quad k = n \end{cases}$$

$$\sigma \;:\; (i_1, \ldots, i_n) \to (1, \ldots, n)$$
$$\tau \;:\; (j_1, \ldots, j_n) \to (1, \ldots, n).$$

Proposition.

$$\left\langle \varphi(a^*),\, \psi(a) \right\rangle \;=\; \int \exp\left(\sum_i a_i^* a_i \right) \varphi(a^*) \; \psi(a) \; da^* \, da \, .$$

Proof. Exercise.

Proposition. *Let $A : V \to V$ such that*

$$\psi_A(a) = \sum_{\{i\}_<,\{j\}_<} a_{\{i\}_<} A_{\{i\}_<\{j\}_<} \psi_{\{j\}_<}$$
$$= \psi_0 \oplus A\psi_1 \oplus A^{\otimes 2}\psi_2 \oplus \cdots .$$

Then

$$\int \exp(-a^* A\, b)\, \exp(-a^* a)\, \psi(a)\, da^*\, da =$$
$$= \psi_A(b) .$$

Proof. Exercise.

Proposition.

$$\int \exp(-a^* A\, b)\, \exp(-a^* a)\, \exp(-B^* B\, a)\, da^* da =$$
$$= \exp(-b^* B\, A\, b) .$$

Remark. This implies $\exp(-b^* A\, b) :=$ "integral kernel" of A acting on $\bigwedge^n V$.

2.2 Vertex operators

(i). Fermionic Fock space: For $\langle V_m \rangle \in \mathbb{C}^{\mathbb{Z}+\frac{1}{2}}$

$$F := \left\{ \begin{matrix} V_{m_1} \wedge V_{m_2} \wedge \cdots & \Big| & m_i \in \mathbb{Z}+\frac{1}{2} \\ m_{i+1} = m_i - 1, \quad i \gg 1 & & \end{matrix} \right\}.$$

(ii). Clifford algebra:

$$Cl_{\mathbb{Z}} := \left\langle \psi_m, \, \psi_m^* \;\middle|\; m \in \mathbb{Z}+\frac{1}{2} \right\rangle$$

$$\psi_m \psi_{m'} + \psi_{m'} \psi_m = \psi_m^* \psi_{m'}^* + \psi_{m'}^* \psi_m^* = 0$$

$$\psi_m \psi_{m'}^* + \psi_{m'}^* \psi_m = \delta_{m\,m'} .$$

(iii). Clifford algebra acts on Fock space F :

$$\psi_m \, v_{m_1} \wedge v_{m_2} \wedge \cdots = v_m \wedge v_{m_1} \wedge v_{m_2} \cdots$$

$$\psi_m^* \, v_{m_1} \wedge v_{m_2} \wedge \cdots = \sum_{i=1}^{\infty} (-1)^i \, \delta_{m_i\,m} \times$$

$$\times \, v_{m_1} \wedge \cdots \wedge \widehat{v_{m_1}} \wedge \cdots$$

(iv). Heisenberg algebra:

$$\left\langle \alpha_n \;\middle|\; n \in \mathbb{Z}\backslash\{0\} \right\rangle : \;\; [\alpha_n, \alpha_{n'}] = -n\, \delta_{n,\,-n'} \,.$$

(v). Heisenberg algebra acts on F :

- As part of Bose-Fermi correspondence in 1D:

$$\alpha_n \longmapsto \sum_{m \,\in\, \mathbb{Z}+\frac{1}{2}} \psi_{m+n}\, \psi^*_m \,.$$

- As operator in F:

$$[\alpha_n, \psi_k] = \psi_{k+n}$$
$$\left[\alpha_n, \psi^*_k\right] = -\,\psi^*_{k-n} \,.$$

(vi). Vertex operators (in F):

$$\Gamma_\pm(x) = \exp\Big(\sum_{n=1}^{\infty} \frac{x^n}{n}\, \alpha_{\pm n}\Big)$$

$$(\Gamma_-(x)\, v,\; w) = (v,\; \Gamma_+(x)\, w) = (\Gamma_+(x)\, w,\; v).$$

(vii). Commutation relations:

$$\begin{aligned}
\Gamma_+(x)\,\Gamma_-(y) &= (1-x) \cdot \Gamma_-(y)\,\Gamma_+(x) \\
\Gamma_+(x)\,\boldsymbol{\psi}(z) &= (1-z^{-1}\,x)^{-1} \cdot \boldsymbol{\psi}(z)\,\Gamma_+(x) \\
\Gamma_-(x)\,\boldsymbol{\psi}(z) &= (1-x\,z)^{-1} \cdot \boldsymbol{\psi}(z)\,\Gamma_-(x) \\
\Gamma_+(x)\,\boldsymbol{\psi}^*(z) &= (1-z^{-1}\,x) \cdot \boldsymbol{\psi}^*(z)\,\Gamma_+(x) \\
\Gamma_-(x)\,\boldsymbol{\psi}^*(z) &= (1-z\,x) \cdot \boldsymbol{\psi}^*(z)\,\Gamma_-(x).
\end{aligned}$$

(viii). Eigenvectors:

$$\Gamma_-(x)\,\prod_i \boldsymbol{\psi}^*(w_i)\,\prod_j \boldsymbol{\psi}^*(z_j)\,V_0^{(n)} =$$

$$= \prod_i (1-x\,z_i)^{-1} \prod_j (1-x\,w_j)\,\prod_i \boldsymbol{\psi}^*(w_i)\,\prod_j \boldsymbol{\psi}^*(z_j)\,V_0^{(n)}$$

where $V_0^{(n)} = v_{n-\frac{1}{2}} \wedge v_{n-\frac{3}{2}} \wedge \cdots$

2.3 Fermionic Kasteleyn operator

$$[Diagram] \qquad [Diagram]$$

where $b(h,t) = \left(h, t-\frac{1}{2}\right)$, and $w(h,t) = \left(h, t+\frac{1}{2}\right)$.

The K-matrix over (b,w), using above-chosen diagram $b \sim w$ is:

$$K(h,t) \;=\; (h,t) - \left(h+\frac{1}{2},\, t+1\right) + x_{h,t}\left(h-\frac{1}{2},\, t+1\right).$$

$$[Diagram]$$

Placing fermions $a^*_{h,\,t}$ and $a_{h,\,t}$, respectively, at $b(h,t)$ and $w(h,t)$, then

$$a^*Ka = \sum_{h,t} a^*_{h,t}\, a_{h,t} - \sum_{h,t} a^*_{h+\frac{1}{2},\,t+1}\, a_{h,t} + \sum_{h,t} a^*_{h-\frac{1}{2},\,t+1}\, a_{h,t}\, x_{h,t} =$$

$$= \sum_t \left(a^*_t a_t + a_t V a^*_{t+1} + a_t V^{-1} x_t\, a^*_{t+1}\right)$$

in addition to considering boundary conditions

$[Diagram]$

Prob $(\pi) \propto$

$\propto \prod_t q_t^{|\pi(t)|}$

$\left(\begin{array}{c}\text{in previous}\\ \text{notations}\\ q_{h,t} = q_t\end{array}\right)$

where the assumption is that $x_{h,t} = x_t$.

Theorem. *Write*

$$Z \;=\; \int \exp\bigl(a^* A\, a\bigr)\, da^*\, da \;=$$

$$= \Bigl\langle \Gamma_-\bigl(x_{-\frac{1}{2}}\bigr) \cdots \Gamma_-\bigl(x_{u_0+\frac{1}{2}}\bigr)\, \Gamma_+\bigl(x_{\frac{1}{2}}\bigr) \cdots \Gamma_+\bigl(x_{u_1+\frac{1}{2}}\bigr)\, V_0^{(0)}\,,\quad V_0^{(0)} \Bigr\rangle\,.$$

Proof. Outline:

$$\int \cdots\; \exp\bigl(a^*_{t-1}\, a_{t-1}\bigr) \cdot \exp\bigl(a_{t-1}\,(V - V^{-1} X_t)\, a^*_t\bigr) \cdot$$

$$\cdot\; \exp\bigl(a^*_t\, a_t\bigr) \cdot \exp\bigl(a_t\,(V - V^{-1} X_t)\, a^*_{t+1}\bigr) \cdots \;=$$

$$= \;\cdots\; \underbrace{\widetilde{\bigl(V - V^{-1} X_{t-1}\bigr)}^{\,-1}}_{\nu_{t-1}} \cdot \underbrace{\widetilde{\bigl(V - V^{-1} X_t\bigr)}^{\,-1}}_{\nu_t} \;\cdots$$

where $\Gamma_+(x_t) = \nu_{t-1}$ or $\Gamma_-(x_t) = \nu_t$ depending on t, $\widetilde{A} := A$ such that $V \hookleftarrow$ is lifted to $\bigwedge^{\frac{\infty}{2}} V$, given by $V := \bigoplus_{m \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\, v_h$, with boundary conditions, etc.

Remark. Direct proof exists combinatorially, without K-orientation formula.

Remark. Write

$$Z = \prod_{m=\frac{1}{2}}^{u_1-\frac{1}{2}} \prod_{m'=u_0+\frac{1}{2}}^{-\frac{1}{2}} \left(1-x^-_{m'}\,x^+_m\right)^{-1}.$$

Theorem. (Okounkov & R., 2005).

$$\left\langle \sigma_{(h_1 t_1)} \cdots \sigma_{(h_k t_k)} \right\rangle = \det(K((t_i,h_i),(t_j,h_j)))_{1\leqslant i,\,j\leqslant k}$$

$$K((t_i,h_i),(t_j,h_j)) =$$

$$= \frac{1}{(2\pi i)^2} \int\limits_{|z|<R(t_1)} \int\limits_{|z|<\widetilde{R}(t_2)} \frac{\Phi_-(z,t_1) \quad \Phi_+(w,t_2)}{\Phi_+(z,t_1) \quad \Phi_-(w,t_2)} \cdot$$

$$\cdot \frac{1}{z-w} \cdot z^{\left(-h_1 - B(t_1) - \frac{1}{2}\right)} \cdot w^{\left(h_2 - B(t_2) - \frac{1}{2}\right)} \; dz\, dw$$

where:

$\lvert w\rvert<\lvert z\rvert,\ t_1\geqslant t_2$	$R(t)=\min\limits_{m>t}\left((x^+_m)^{-1}\right),\ \widetilde{R}(t)=\max\limits_{m<t}(x^-_m),\ B(t)=\frac{\lvert t\rvert}{2}-\frac{\lvert t-u_0\rvert}{2},$
$\lvert w\rvert>\lvert z\rvert,\ t_1<t_2$	$\Phi_+(z,t)=\prod\limits_{m>\max\left(t,\frac{1}{2}\right)}(1-z\,x^+_m),\ \Phi_-(z,t)=\prod\limits_{m<\max\left(t,-\frac{1}{2}\right)}\left(1-z^{-1}\,x^-_m\right).$

2.4 Thermodynamic limit with scaling

$$[Diagram] \qquad \left.\begin{array}{l} x_m^+ = aq^m \\ x_m^- = a^{-1}q^m \end{array}\right\} \text{assumed}$$

corresponds to $\text{Prob}(\pi) \propto q^{|\pi|}$.

2.5 Asymptotics of partition functions

Consider limit $\varepsilon \to 0$, $q = e^{-\varepsilon}$, $u_1 = \varepsilon^{-1} v_1$, $u_0 = \varepsilon^{-1} v_0$, where v_1, v_0 are fixed:

$$Z = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} \left(1 - x_m^- x_n^+\right)^{-1} = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} \left(1 - q^{m-n}\right)^{-1}$$

$$\langle |\pi| \rangle = q \frac{\partial}{\partial q} \ln Z$$

$$\ln Z = \varepsilon^{-2} \int_0^{u_1} \int_{u_0}^{0} \ln \Big(\underbrace{1 - e^{-s+t}}_{\text{2D partition function}} \Big) \, ds \, dt + \cdots$$

$$\langle |\pi| \rangle = \varepsilon^{-3} \int_0^{u_1} \int_{u_0}^{0} \underbrace{\frac{s-t}{1 - e^{t-s}}}_{\text{3D volume function}} \, ds \, dt + \cdots$$

Consider limit $\varepsilon \to 0$ for $t_i = \varepsilon^{-1}\tau_i$, $h_1 = \varepsilon^{-1}\chi_i$, where τ_i, χ_i are fixed:

$[Diagram]$ $(\tau_i,\, \chi_i)$ *in the bulk*

$$K((t_1,h_1),(t_2,h_2)) \; = \; \frac{1}{(2\pi i)^2}\int_{C_z}\int_{C_w} \exp\!\Big(\varepsilon^{-1}\left(S(z,t_1,\chi_1) - S(z,t_2,\chi_2)\right)\Big) \cdot \cdot\; (zw)^{1/2}\,(z-w)^{-1}\;dz\,dw\,.$$

$$S(z,t,\chi) \; = \; -\Big(\chi+\frac{\tau}{2}-u_0\Big)\ln Z + {} + \mathsf{Li}_2\!\left(ze^{-v_0}\right) + \mathsf{Li}_2\!\left(ze^{-v_1}\right) - \mathsf{Li}_2(z) - \mathsf{Li}_2\!\left(ze^{-\tau}\right)$$

where

$$\mathsf{Li}_2(z) \; = \; \int_0^z t^{-1}\ln(1-t)\,dt\,.$$

2.6 Critical points of $S(z)$

$$\exp\left(\chi + \frac{\tau}{2}\right) = \frac{\left(1 - ze^{-v_0}\right)\left(1 - ze^{-v_1}\right)}{(1 - z)\left(1 - ze^{-\tau}\right)}$$

gives quadratic equation. That is, 2 real solutions, or 2 complex conjugate, or zero discriminant.

$[Diagram]$

$$\boxed{\partial_\chi h_0(\tau, \chi) = \frac{1}{\pi}\,\mathrm{arg}(z_0)}$$

$$\left\langle \sigma_{(h,t)} \right\rangle = K((t,h),(t,h)) \to \varepsilon\, \partial_\chi h_0(\tau, \chi)$$

2.7 Results of steepest descent

$$K((t_1,h_1),(t_2,h_2)) \;=\; -\frac{\varepsilon}{2\pi}\cdot\Bigg(\frac{\exp\big(\varepsilon^{-1}(S_1(z_1)-S_2(w_2))\big)}{(z_1-w_2)\,\sqrt{-w_2S_2''(w_2)}\,\sqrt{z_1S_1''(z_1)}} -$$

$$-\frac{\exp\big(\varepsilon^{-1}(S_1(z_1)-S_2(\overline{w}_2))\big)}{(z_1-\overline{w}_2)\,\sqrt{-\overline{w}_2S_2''(\overline{w}_2)}\,\sqrt{z_1S_1''(z_1)}} + c.\,c.\Bigg)\cdot\,(1+O(1))$$

where $z_0(\chi,\tau) :=$ inner of limit shape $\mathcal{H}_+ := \{z\in\mathbb{C}, \mathrm{Im}\, z>0\}$ such that

$$z_1 := z_0(\chi_1,\tau_1),\;\; w_2 := z_0(\chi,\tau).$$

Therefore,

$$K((t_1,h_1),(t_2,h_2)) = \frac{\varepsilon}{2\pi}\exp\big(\varepsilon^{-1}(\mathrm{Re}(S(z_0(\chi_1,\tau_1)))-\mathrm{Re}(S(z_0(\chi_2,\tau_2))))\big)\,\cdot$$

$$\cdot\Bigg(\frac{\exp\big(i\varepsilon^{-1}(\mathrm{Im}(S'(z_1))-\mathrm{Im}(S(w_2)))\big)}{(z_1-w_2)}+\frac{\exp\big(i\varepsilon^{-1}(\mathrm{Im}(S'(z_1))-\mathrm{Im}(S(\overline{w}_2)))\big)}{(z_1-\overline{w}_2)}$$

$$+\;c.\,c.\Bigg)\cdot\,(1+O(1))\quad(*)\,.$$

This suggests convergence of K-orientation fermions to free Dirac fermions:

$$\frac{1}{\sqrt{\varepsilon}}\psi_{\vec{x}} = \exp\big(\varepsilon^{-1}\mathsf{Re}(S(z_0))\big)\cdot\Big(\psi_{+}(z_0)\,\exp\big(i\varepsilon^{-1}\mathsf{Im}(S(z_0))\big) + $$
$$+\ \psi_{-}(\overline{z}_0)\,\exp\big(i\varepsilon^{-1}\mathsf{Im}(S(z_0))\big)\Big)\cdot(1+O(1))$$

$$\frac{1}{\sqrt{\varepsilon}}\psi^{*}_{\vec{x}} = \exp\big(\varepsilon^{-1}\mathsf{Re}(S(z_0))\big)\cdot\Big(\psi^{*}_{+}(z_0)\,\exp\big(i\varepsilon^{-1}\mathsf{Im}(S(z_0))\big) + $$
$$+\ \psi^{*}_{-}(\overline{z}_0)\,\exp\big(i\varepsilon^{-1}\mathsf{Im}(S(z_0))\big)\Big)\cdot(1+O(1))$$

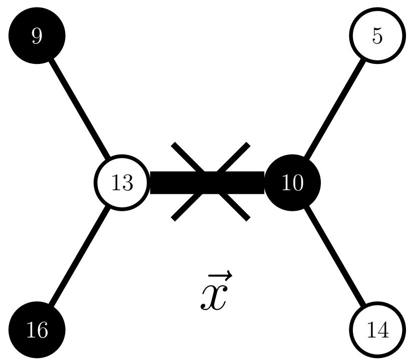
where

$$\mathbb{E}\big(\psi^{*}_{\pm}(z)\ \psi_{\pm}(w)\big) = \frac{1}{z-w}$$
$$\mathbb{E}\big(\psi^{*}_{\pm}(z)\ \psi_{\mp}(w)\big) = \mathbb{E}\big(\psi^{*}\,\psi^{*}\big) = \mathbb{E}(\psi\,\psi) = 0$$

$\psi^{*}_{\pm}(z)$, $\psi_{\pm}(w)$ are spinors:

$$\psi^{*}_{\pm}(z) = \psi^{*}_{\pm}(w)\sqrt{\frac{\partial w}{\partial z}}$$
$$\psi_{\pm}(z) = \psi_{\pm}(w)\sqrt{\frac{\partial w}{\partial z}}\,.$$

The correlation functions are derived as follows:



$$\Big\langle (\sigma_{\vec{x}_1} - \langle \sigma_{\vec{x}_1} \rangle)\ (\sigma_{\vec{x}_2} - \langle \sigma_{\vec{x}_2} \rangle) \Big\rangle = K_{12} K_{21} =$$

$$= \frac{\varepsilon^2}{(2\pi)^2} \left(\frac{\frac{\partial z_1}{\partial x_1}\ \frac{\partial w_2}{\partial x_2}}{(z_1 - w_2)^2} \ - \ \frac{\frac{\partial z_1}{\partial x_1}\ \frac{\partial \overline{w}_2}{\partial x_2}}{(z_1 - \overline{w}_2)^2} \ + \ c.\,c. \right) \times$$

$$\times\,(1 + O(1)).$$

That is, correlation functions are given by

$$\sigma_{\vec{x}_1} - \langle \sigma_{\vec{x}_1} \rangle \ = \ \varepsilon\,\partial_x\,\varphi(z_0(\tau, x)) + \cdots$$

where $\varphi(z) :=$ Gaussian free field on $\mathcal{H}_+$.

The Green's function of the Dirichlet problem on $\mathcal{H}_+$ is given by

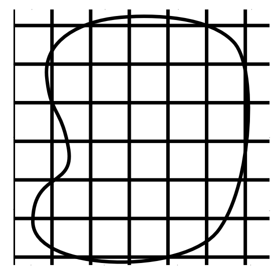
$$\langle \varphi(z)\,\varphi(w) \rangle \ = \ \frac{1}{2\pi}\,\ln \left| \frac{z - w}{z - \overline{w}} \right|.$$

And, the Bose-Fermi correspondence is given by

$$\partial_x \varphi \ = \ :\widetilde{\psi}(z, \overline{z})\ \widetilde{\psi}(z, \overline{z}): \ \cdots \ .$$

2.8 Scaling limit in the Kasteleyn operator

For lattice L such that $\Gamma = D_\varepsilon := \varphi_\varepsilon(L) \cap D$



Let $\Sigma_\Gamma^K :=$ difference operator. Then

$$\left(\Sigma_\Gamma^K\right)_x \cdot G_{x,y} = \delta_{x,y}$$

where $\varepsilon \to 0$ in asymptotic for $G_{x,y}$.

Particular cases.

(i). Hexagonal lattices, with weights as above, such that

$$q_t = e^{-\varepsilon f(t)}, \ t = \frac{\tau}{\varepsilon}, \ \varepsilon \to 0.$$

Theorem. $G_{x,y} :=$ *same as* $(\star)$, *with different* $z_0(\tau, x)$.

(ii). Periodic lattices.

2.9 Limit shapes and variational principle

(i). For the $N \times M$ torus

$$[Diagram]$$

$$
\begin{aligned}
Z(H,V) &= \sum_{D} \prod_{\ell} \omega(\ell)\, \exp(H\Delta_a h_D + V\Delta_b h_D) \\
&= \frac{1}{2}\Big(\mathsf{Pf}\big(A^{K_1}\big) + \mathsf{Pf}\big(A^{K_2}\big) + \mathsf{Pf}\big(A^{K_3}\big) - \mathsf{Pf}\big(A^{K_4}\big)\Big)
\end{aligned}
$$

where $N, M \to \infty$, $\dfrac{N}{M} :=$ fixed.

And, $\omega(\ell) = 1$ gives Kasteleyn matrices' eigenvalues by Fourier transform.

Theorem. (McCoy & Wu, 1969; Kenyon & Okounkov, 2005).

$$
\begin{aligned}
\lim_{N,M\to\infty} \frac{1}{NM} \ln Z_{NM} &= \oint\oint \ln|1+zw|\, \frac{dz}{z}\frac{dw}{w} \\
&= f(H,V) := \begin{cases} |z| = e^H \\ |w| = e^V \end{cases}.
\end{aligned}
$$

(ii). Taking Legendre transform

$$\sigma(s,\, t) \;=\; \max_{H,V} \left(H_s + V_t - f(H,\, V)\right)$$

then

$$\sum_D 1 \;=\; \sum_D \prod_D w\,(e) \;=\; \exp(NM\,\sigma(s,\, t)\cdot(1+O(1)))$$

where

$$\frac{\Delta_a h_D}{N} = s, \quad \frac{\Delta_b h_D}{M} = t, \quad N, M \to \infty, \quad \frac{N}{M} \text{ fixed.}$$

(iii). For the domain

$$[Diagram]$$

$\Delta_a h = sN,\ \Delta_b h = tM.$

Theorem. (Cohn, Kenyon, & Propp, 2000). *As a result,*

$$\sum_D 1 \;=\; \exp(NM\,\sigma(s,\, t)\cdot(1+O(1)))$$

with those boundary conditions of height function h_D.

(iv). For the domain

$$[Diagram] \qquad\qquad N_i \times M_j$$

$$Z_{D_{\boldsymbol{\varepsilon}}} = \sum_{\left\{\substack{\text{values of}\\ \text{height functions}\\ \text{on}\\ \text{boundaries}\\ \text{between rectangles}}\right\}} Z_{N_i \underset{M_j}{\square}} (h_{\text{bound}})$$

$$= \sum_{\{\Delta_x h,\, \Delta_y h\}_{ij}} \exp\left(\sum_{N_i \underset{M_j}{\square}} N_i\, M_j\ \boldsymbol{\sigma}\left(\frac{\Delta_x h}{M_j},\, \frac{\Delta_y h}{N_i}\right) \right)$$

$$= \exp\left(\boldsymbol{\varepsilon}^{-2} \int_D \boldsymbol{\sigma}\left(\partial_x h_0,\, \partial_y h_0\right)\, dx\, dy\ \left(1 + O\left(1\right)\right) \right)$$

where $h_0 :=$ minimizer for

$$S\left[h\right] = \int_D \boldsymbol{\sigma}\left(\partial_x h_0,\, \partial_y h_0\right)\, dx\, dy\,.$$

Theorem. (Cohn, Kenyon, & Propp, 2000). *In addition,*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln Z_{D_\varepsilon} = \int_D \sigma(\overrightarrow{\nabla} h_0) dx\, dy$$

such that

- h_0 *is a minimizer*
- $0 < \partial_x h, \partial_y h < 1$
- $h_0\big|_{\partial D} = b$, *the boundary condition appearing in the limit* $\varepsilon \to 0$.

$$[Diagram]$$

for height function

$$h = \varepsilon^{-1} h_0 + \varphi = \varepsilon^{-1}(h_0 + \varepsilon\varphi)$$

where $h_0 :=$ *the limit shape, and* $\varphi :=$ *fluctuations.*

2.10 Physics way of describing fluctuations

$$S[h_0+\varepsilon\varphi] = S[h_0] + \frac{\varepsilon^2}{2}\iint_D a^{ij}(x)\partial_i\varphi\,\partial_j\varphi\,d^2x$$

$$a^{ij}(x) = \partial_i\partial_j\,\varphi(s,t)\begin{cases} s=\partial_1 h_0 \\ t=\partial_2 h_0\end{cases}$$

Partition function is given by

$$Z = \exp(\varepsilon^{-2}S(h_0))\int \exp\left(\frac{1}{2}\iint_D a^{ij}(x)\partial_i\varphi\,\partial_j\varphi\,d^2x\right)D\varphi$$

where D := scalar field with Riemannian metric induced by h_0.

Correlation functions are given by

$$\langle\varphi(x)\,\varphi(y)\rangle = G(x,y)$$

where G := Green's function for $\Delta=\partial_i(a^{ij}\partial_j)$.

Conjecture. G := *same as from asymptotics of Kasteleyn operators.*

Remark. The conjecture := theorem always in certain cases (R., et. al.).

2.11 Conclusion - limit topology phenomena, ongoing

1. How to make such pictures of (i.e. simulate) random configuration:

   (i). Monte Carlo for $\exp\left(\propto 1000^2\right)$

   (ii). Sampling around most probable region

   (iii). Markov Chain Monte Carlo method.

2. How to describe limit topologies and fluctuations analytically:

   (i). Kasteleyn matrix solution and correlation function

   (ii). Variational principle: Minimizing large deviation functionals

   (iii). Boundary conditions.

Thank You!