Static and dynamics equilibria in binary choice games on graphs

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- Consider a graph G with nodes equipped with binary variables $\sigma_i = \pm 1, \; i = 1, \cdots, N$
- A state of a system is fully characterized by the multivariate probability distribution

$$\mathcal{P}\left(\bar{\sigma}(t)\right) = \mathcal{P}\left(\sigma_1(t), \sigma_2(t), \cdots, \sigma_N(t)\right),$$

  where $\mathcal{P}\left(\bar{\sigma}(t)\right)$ is a probability of having at some time moment t some particular cofiguration $\bar{\sigma}$
- The multivariate distribution $\mathcal{P}\left(\bar{\sigma}(t)\right)$ obeys normalization condition

$$\sum_{\bar{\sigma}(t)} \mathcal{P}\left(\bar{\sigma}(t)\right) = 1$$

- With distribution $\mathcal{P}$ one can calculate all possible quantitative characteristics of a system, e.g.
  - mean values

$$\langle \sigma_i(t) \rangle = \sum_{\bar{\sigma}(t)} \sigma_i(t) \mathcal{P}\left(\bar{\sigma}(t)\right)$$

  - correlations

$$\langle \sigma_i(t) \sigma_i(t) \rangle = \sum_{\bar{\sigma}(t)} \sigma_i(t) \sigma_j(t) \mathcal{P}\left(\bar{\sigma}(t)\right)$$

- An equilibrium is naturally defined as a state in which $\mathcal{P}\left(\bar{\sigma}(t)\right)$ does not depend on time

$$\frac{d\mathcal{P}\left(\bar{\sigma}(t)\right)}{dt} = 0$$

- Temporal evolution of the system is driven by transitions (choice flip)
$$\sigma_i(t) \to -\sigma_i(t),$$
which are characterized by transition probability per unit time $w(\sigma_i \to -\sigma_i)$.
- Assuming that within a small time interval only one transition is possible we arrive at the evolution equation for $\mathcal{P}(\bar{\sigma}(t))$ (master equation):

$$\begin{aligned} \frac{d\mathcal{P}(\bar{\sigma}, t)}{dt} &= \sum_i [w(-\sigma_i \to \sigma_i)\mathcal{P}(\sigma_1, \cdots, -\sigma_i, \cdots \sigma_N) \\ &\quad - w(\sigma_i \to -\sigma_i)\mathcal{P}(\sigma_1, \cdots, \sigma_i, \cdots \sigma_N)] \end{aligned}$$

- From the master equation evolution equations for fundamental aggregated characteristics of the syste - various correlation functions can be derived:
- Evolution of local mean values:

$$\begin{aligned} \langle \sigma_i \rangle (t) &= \sum_{\bar{\sigma}} \sigma_i \mathcal{P} \left(\bar{\sigma}(t) \right) \\ \frac{d \langle \sigma_i \rangle}{dt} &= -2 \langle \sigma_i w(\sigma_i \to -\sigma_i) \rangle \end{aligned}$$

- Evolution of two-point correlations:

$$\begin{aligned} \langle \sigma_i \sigma_j \rangle (t) &= \sum_{\bar{\sigma}} \sigma_i \sigma_j \mathcal{P} \left(\{\sigma\}, t \right) \\ \frac{d \langle \sigma_i \sigma_j \rangle}{dt} &= -2 \langle \sigma_i \sigma_j \left[w(\sigma_i \to -\sigma_i) + w(\sigma_j \to -\sigma_j) \right] \rangle \end{aligned}$$

Assume that utility of choosing some particular σ_i at time t is a combination of idiosyncratic (including random) factors and externalia due to "interactions"with first neighbors of the node i:

$$U_i(\sigma_i(t)) = \sigma_i \left(H + H_i + \sum_{j \neq i} J_{ij} \, a_{ij} \sigma_j(t) \right) + \epsilon(\sigma_i(t)),$$

where J_{ij} - is a matrix of coupling constants (intensities of the influence of the node j at the node i), a_{ij} - adjacency matrix of the graph G, so that $a_{ij} = 1$ if the link $j \to i$ does exist, and $\epsilon(\sigma_i(t))$ - is some random variabl.

- Opinion change (lfip) $-\sigma_i(t) \to \sigma_i(t)$ is assumed to be a consequence of utility gain

$$-\sigma_i(t) \to \sigma_i(t) \quad \leftrightarrow \quad U(\sigma_i(t)) > U(-\sigma_i(t))$$

- In more details, the flip $-\sigma_i(t) \to \sigma_i(t)$ takes place if

$$2\sigma_i(t) \left(H + H_i + \sum_{j \neq i} J_{ij} \; a_{ij} \sigma_j(t) \right) > \epsilon(-\sigma_i(t)) - \epsilon(\sigma_i(t))$$

- Therefore

$$\mathrm{Prob}(-\sigma_i \to \sigma_i) \sim F^< \left(2\sigma_i \left(2H + H_i + \sum_{j \neq i} J_{ij} \; a_{ij} \sigma_j(t) \right) \right)$$

where $F^<(x) = \mathrm{Prob}\left(\epsilon(-\sigma) - \epsilon(\sigma)\right) < x$

- Let us note, that considering a particular problem of choosing $\pm\sigma_i$ at time t, it is natural to write

$$\mathrm{Prob}(\sigma_i) \equiv P_0(\sigma_i) \quad \sim \quad F^{<}\left(2\sigma_i\left(2H + H_i + \sum_{j\neq i} J_{ij}\; a_{ij}\sigma_j(t)\right)\right)$$

$$\mathrm{Prob}(-\sigma_i) \equiv P_0(-\sigma_i) \quad \sim \quad F^{<}\left(-2\sigma_i\left(2H + H_i + \sum_{j\neq i} J_{ij}\; a_{ij}\sigma_j(t)\right)\right)$$

- In what follows, following BD (2003) we shall use the following parametrization:

$$F^{<}(x) = \frac{1}{1+\exp(-g(x))}$$

- Consider the evolution of the local average opinion:

$$\frac{d\langle\sigma_i\rangle}{dt} = -2\langle\sigma_i w(\sigma_i \to -\sigma_i)\rangle = -2\lambda\langle\sigma_i P_0(-\sigma_i)\rangle$$

- For the binary choice one has a simple equality:

$$P_0(-\sigma_i) = \frac{1}{2}\left[1 - \sigma_i\langle\sigma_i\rangle_0\right],$$

where

$$\langle\sigma_i\rangle_0 = \sigma_i\left(P_0(\sigma_i) - P_0(-\sigma_i)\right)$$

- Therefore

$$\frac{d\langle\sigma_i\rangle}{dt} = -\lambda(\langle\sigma_i(t)\rangle - \langle\sigma_i(t)\rangle_0)$$

- Using the above-introduced parametrization of $F^{<}(x)$ we get

$$\langle \sigma_i \rangle_0 = \left\langle \tanh\left[\frac{1}{2} g \left(2H + 2H_i + 2\sum_{j\neq i} J_{ij}\; a_{ij}\sigma_j(t)\right)\right]\right\rangle_{-i}$$

- Averaging in the rhs includes correlation functions of all orders:

$$\begin{aligned}\langle \sigma_i \rangle_0 &= \frac{1}{2}\left\langle g\left(2H + 2H_i + 2\sum_{j\neq i} J_{ij}\; a_{ij}\sigma_j(t)\right)\right\rangle_{-i} \\ &- \frac{1}{12}\left\langle g^3\left(2H + 2H_i + 2\sum_{j\neq i} J_{ij}\; a_{ij}\sigma_j(t)\right)\right\rangle_{-i} + \cdots\end{aligned}$$

- Manageable expressions can be obtained by truncating the coupled equations for correlations at some level : mean field $\langle f(x)\rangle \simeq f(\langle x\rangle)$, binary correlations, etc.

- Consider the simplest case of the "Ising model where $J_{ij} = J > 0$, $H_i = 0$
- Consider temporal evolution of the average choice

$$m(t) = \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i \rangle$$

- For the simplest topology of complete graph we get, in the mean field approximation

$$\frac{dm(t)}{dt} = -\lambda \left(m(t) - \tanh \left(\frac{1}{2} [2H + 2Jm(t)] \right) \right)$$

which coinsides with the result of BD (2003).

- Therefore, equilibria are characterized by the generalized Curie-Weiss equations

$$m = \tanh \left(\frac{1}{2} g [2H + 2Jm] \right)$$

- For arbitrary graph the simplest (and at present the only existent) approximation allowing to take into account the degree distribution p_k corresponds to the replacement

$$a_{ij} \to \frac{k_i k_j}{\langle k \rangle N}$$

- A natural choice of the order parameter is

$$m_w = \frac{1}{\langle k \rangle} \frac{1}{N} \sum_{j \neq i} k_j \langle \sigma_j \rangle$$

- In the mean field approximation

$$\frac{dm_w}{dt} = -\left(m_w - \sum \frac{kp_k}{\langle k \rangle} \tanh\left(\frac{1}{2} g(2H + 2Jkm_w) \right) \right)$$

- Equilibria are thus characterized by the following equation

$$m_w = \sum \frac{kp_k}{\langle k \rangle} \tanh\left(\frac{1}{2} g(2H + 2Jkm_w) \right)$$

Заключение

- A simplest dynamics of the choices of myopic agents on graphs using the noisy best response is formulated.
- It is shown that existing results can be reproduced in the simplest approximation.
- The presented consideration shows that game-theoretic constructions used for the analysis of static equilibria (mixed strategies, discrete response equilibrium) can be considered as simplifications of a more general game-theoretic formalism taking into account correlation functions of all orders.