On the Metric Dimension of Cartesian Powers of a Graph

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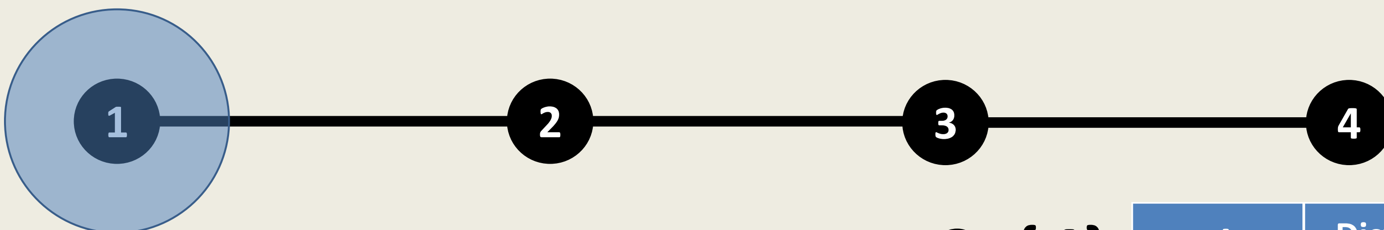
Outline:

- Introduction
- Related questions
- Results
- Open problems

Resolving set

Set $\boldsymbol{S} \subseteq \boldsymbol{V}(\boldsymbol{G})$ resolves graph $\boldsymbol{G}$ if any vertex $\boldsymbol{x} \in \boldsymbol{V}(\boldsymbol{G})$ is uniquely determined by its distance vector $(\boldsymbol{d}(\boldsymbol{x},\boldsymbol{y}))_{\boldsymbol{y}}$, where $\boldsymbol{y} \in \boldsymbol{S}$

Example:



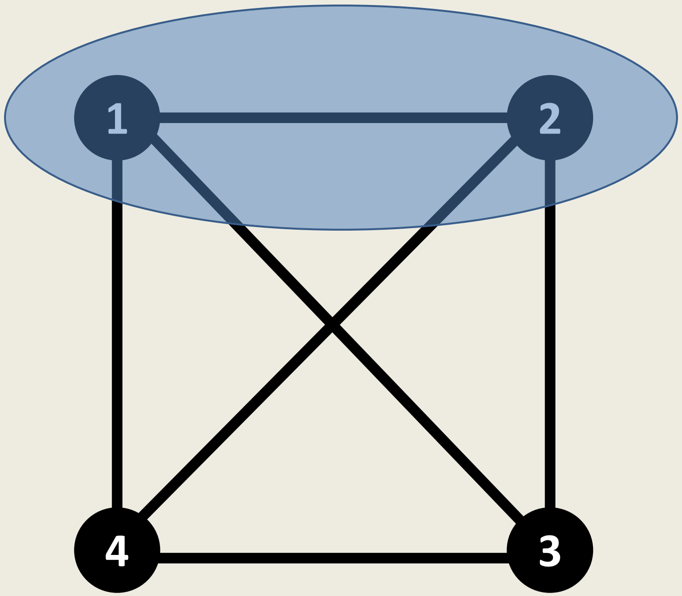
$\boldsymbol{S}=\{\boldsymbol{1}\}$

vertex	Distance vector
1	($\boldsymbol{0}$)
2	($\boldsymbol{1}$)
3	($\boldsymbol{2}$)
4	($\boldsymbol{3}$)

Map between vertices and distance vectors is injective => $\boldsymbol{S}=\{\boldsymbol{1}\}$ resolves $\boldsymbol{P_4}$

Resolving set

Example:



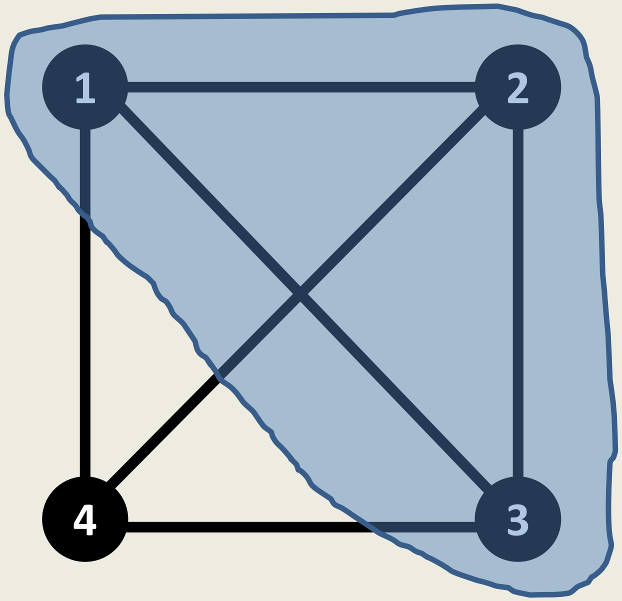
S={***1***,***2***}

vertex	Distance vector
1	(***0***,***1***)
2	(***1***,***0***)
3	(***1***,***1***)
4	(***1***,***1***)

Map between vertices and distance vectors is **not** injective **=>** ***S***={***1***,***2***} does not resolve K_4

Resolving set

Example:



S={***1***,***2***,***3***}

vertex	Distance vector
1	(***0***,***1***,***1***)
2	(***1***,***0***,***1***)
3	(***1***,***1***,***0***)
4	(***1***,***1***,***1***)

Map between vertices and distance vectors is injective **=>** ***S***={***1***,***2***,***3***} is a resolving set of K_4

Metric dimension of a graph

Given graph $\boldsymbol{G}$ define the metric dimension of $\boldsymbol{G}$:

$\boldsymbol{m}(\boldsymbol{G}) = \min|\boldsymbol{S}|$, where $\boldsymbol{S}$ is a resolving set of $\boldsymbol{G}$

Example:

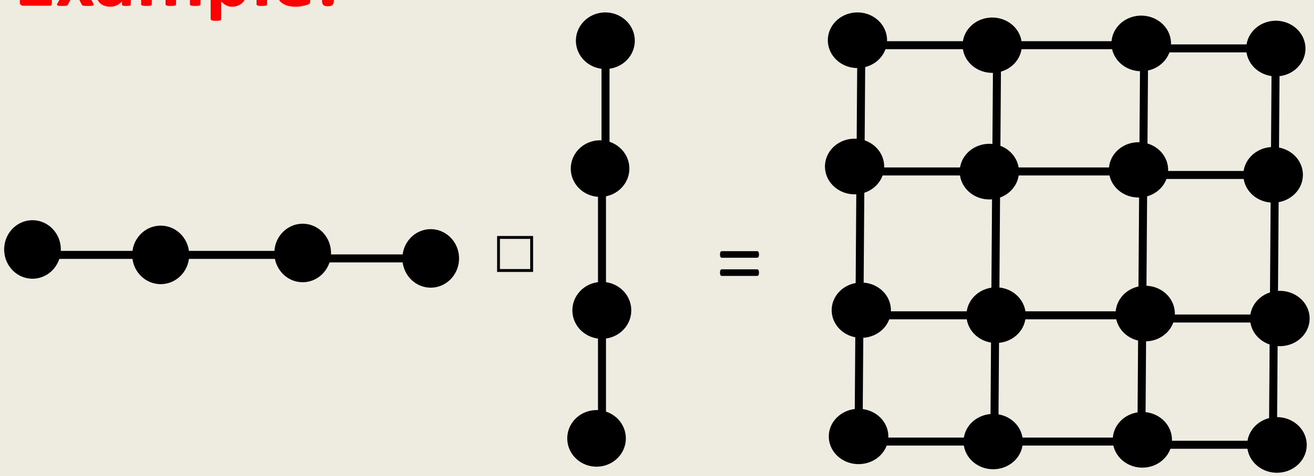
$$\boldsymbol{m}(\boldsymbol{P_4}) = \boldsymbol{1};$$
$$\boldsymbol{m}(\boldsymbol{C_4}) = \boldsymbol{2};$$
$$\boldsymbol{m}(\boldsymbol{K_4}) = \boldsymbol{3};$$

Cartesian power

Given graph G define the Cartesian power $G^{\square n}$

- **Vertex set:** $\underbrace{V(G) \times \ldots \times V(G)}_{n \text{ times}}$
- **Edge set:** $(u_1,\ldots,u_n) \sim (v_1,\ldots,v_n)$ iff $\begin{array}{ccccc} (u_1, & \ldots, & u_i, & \ldots, & u_n) \\ \parallel & \cdots & \wr & \cdots & \parallel \\ (v_1, & \ldots, & v_i, & \ldots, & v_n) \end{array}$

Example:



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$$\begin{array}{ccccc} (u_1, & \ldots, & u_i, & \ldots, & u_n) \\ \shortparallel & \cdots & \wr & \cdots & \shortparallel \\ (v_1, & \ldots, & v_i, & \ldots, & v_n) \end{array}$$

Example:

$K_2^{\square n}$ is the n-dim. Boolean hypercube

$K_q^{\square n}$ is the n-dim. q-ary Hamming space

Metric dimension of Cartesian power

Given graph ***G*** and integer ***n*** define

$$m(G,n) \triangleq m(G^{\square n})$$

Problem:

find the asymptotic behavior of $m(G,n)$ as $n \to \infty$

Metric dimension of Cartesian power

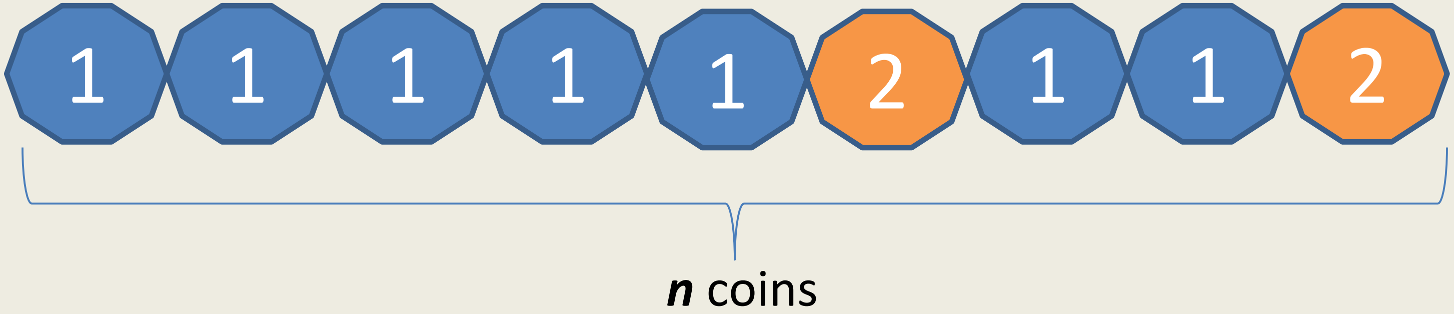
Examples:

G = $\boldsymbol{K_2}$ (coin weighing problem)

G = $\boldsymbol{K_q}$ (mastermind game)

Coin weighing problem

Suppose ***n*** coins of weight ***a*** or ***b*** are given, where ***a*** and ***b*** are known.

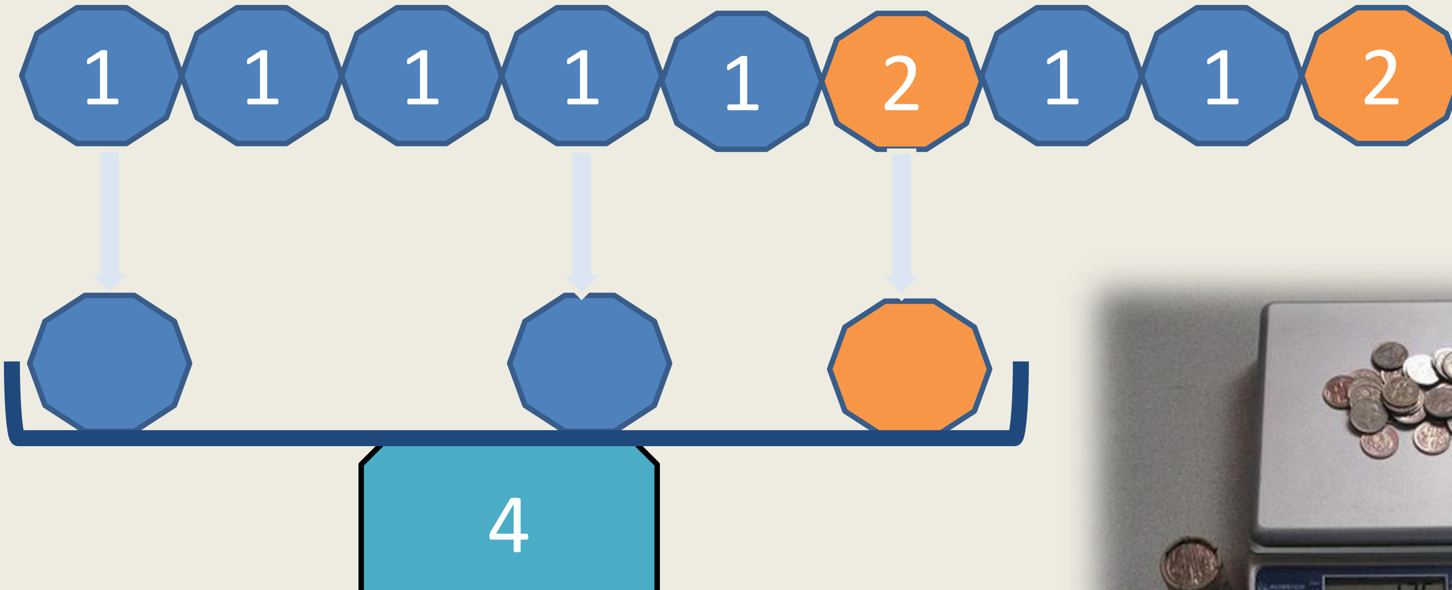


Goal:

identify which of ***n*** coins are of weight ***a***

Coin weighing problem

Suppose ***n*** coins of weight ***a*** or ***b*** are given, where ***a*** and ***b*** are known.



What we can do:

Take an arbitrary subset of coins and weigh it using digital accurate scale.

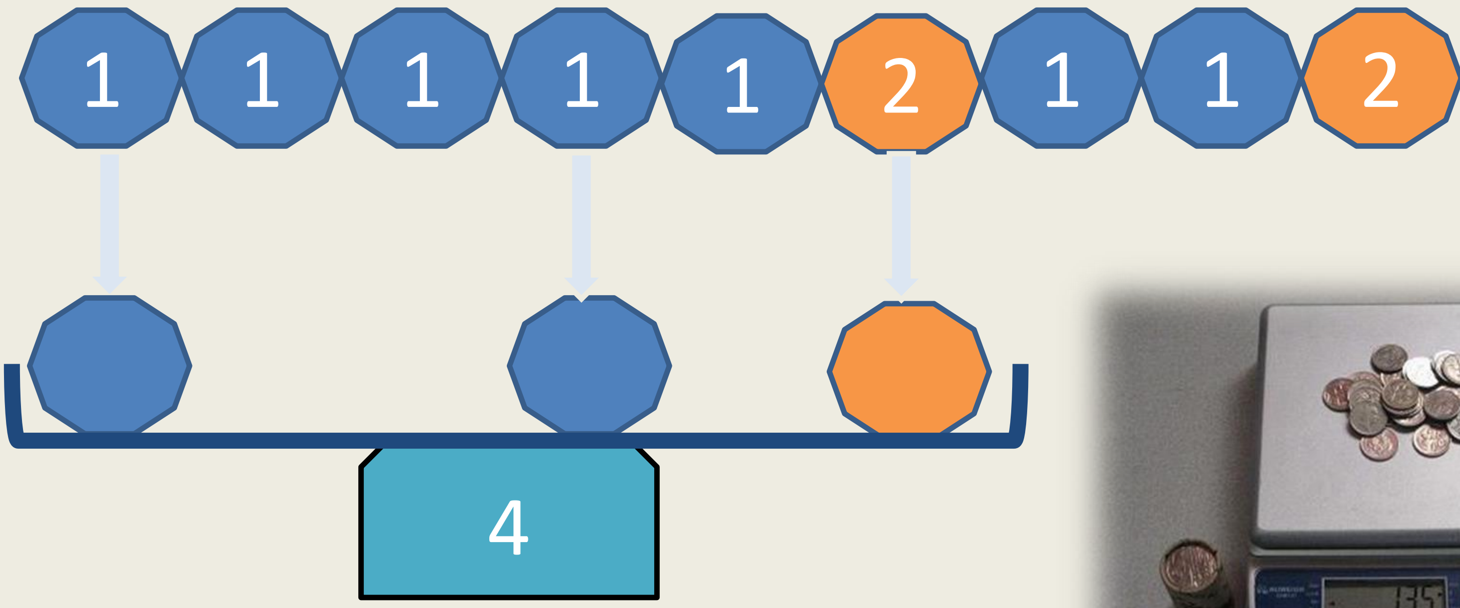
Coin weighing problem

Method:
design group tests of coins s.t. any ***n*** coins of weight ***a*** or ***b*** can be uniquely determined by group weighings.

Problem:
Given ***n*** minimize the number ***cw***(***n***) of weighings

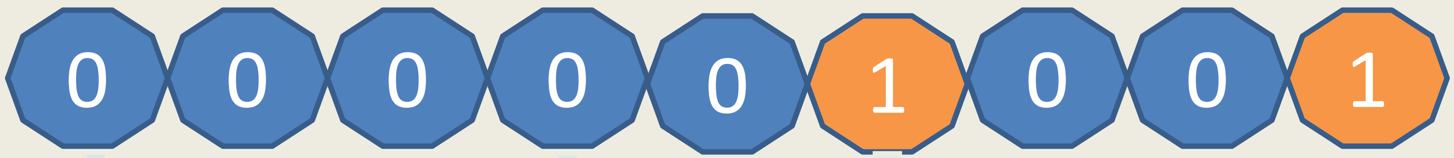
Coin weighing problem

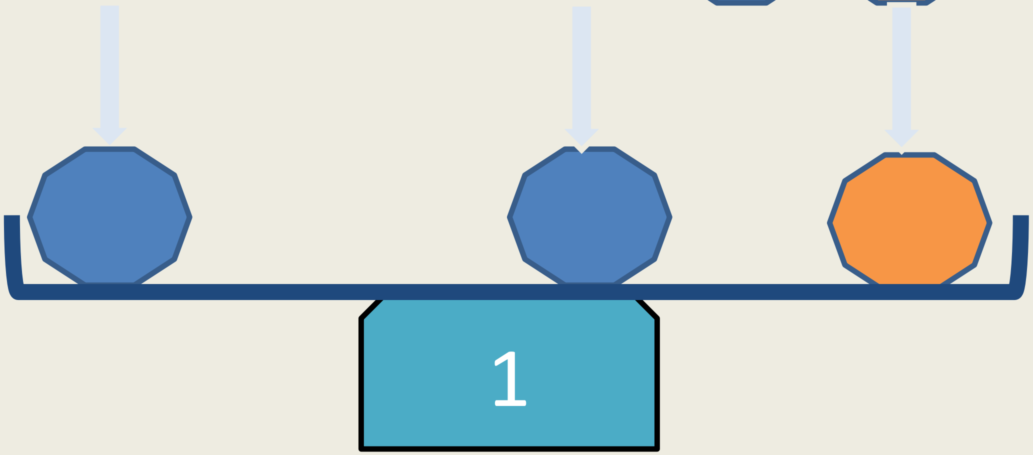
Suppose ***n*** coins of weight ***a*** or ***b*** are given, where ***a*** and ***b*** are known.



Coin weighing problem

Suppose ***n*** coins of weight ***0*** or ***1*** are given

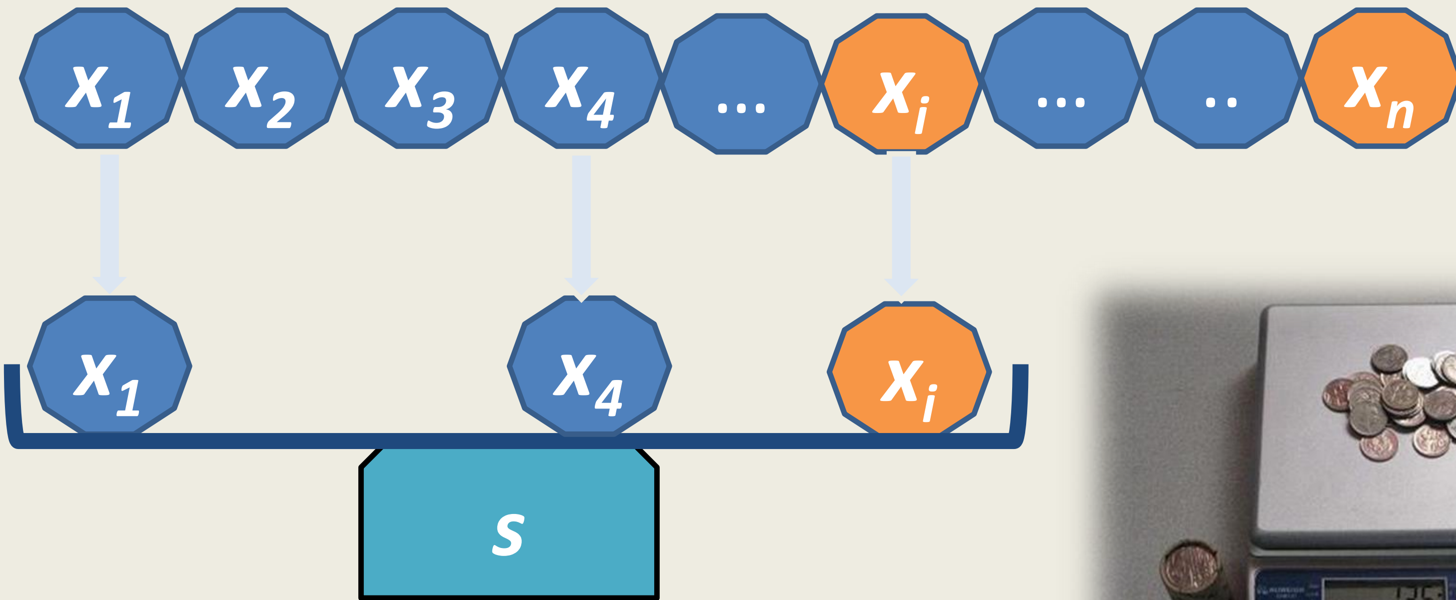




WLOG: a = 0, b = 1

Coin weighing problem

Suppose ***n*** coins of weight ***0*** or ***1*** are given



Weight of coins
with indices $I \subset [n]$ is s $\Longleftrightarrow$ $\sum_{i \in I} x_i = s$

Coin weighing problem

Suppose ***n*** coins of weight ***0*** or ***1*** are given

$$\langle \mathbf{x}, \boldsymbol{y} \rangle = \boldsymbol{s}, \quad \text{where } \boldsymbol{y_i} = \begin{cases} \boldsymbol{1}, & i \in \boldsymbol{I} \\ \boldsymbol{0}, & \text{o/w} \end{cases}$$

⇕

Weight of coins with indices $\boldsymbol{I} \subset [\boldsymbol{n}]$ is $\boldsymbol{s}$ $\Longleftrightarrow$ $\sum_{i \in \boldsymbol{I}} \boldsymbol{x_i} = \boldsymbol{s}$

Coin weighing problem

Equivalently, any $\boldsymbol{x} \in \{\boldsymbol{0,1}\}^n$ is uniquely determined by its **inner product vector** $(<\boldsymbol{x},\boldsymbol{y}>)_{\boldsymbol{y}}$, where $\boldsymbol{y} \in \boldsymbol{S} \subseteq \{\boldsymbol{0,1}\}^n$

$$<\boldsymbol{x},\boldsymbol{y}> \; = \boldsymbol{t}_{1,1}(\boldsymbol{x},\boldsymbol{y})$$

$$\boldsymbol{x} = (1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1)$$

$$\boldsymbol{y} = (0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0)$$

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$$\langle \boldsymbol{x},\boldsymbol{y} \rangle = \boldsymbol{t}_{1,1}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{n} - \boldsymbol{t}_{0,1}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{t}_{1,0}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{t}_{0,0}(\boldsymbol{x},\boldsymbol{y})$$

$$\boldsymbol{x} = (1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1)$$

$$\boldsymbol{y} = (0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0)$$

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$$\boldsymbol{x = (1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1)}$$

$$\boldsymbol{y = (0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0)}$$

Coin weighing problem

Equivalently, any $\boldsymbol{x} \in \{\boldsymbol{0,1}\}^n$ is uniquely determined by its **inner product vector** $(\langle \boldsymbol{x},\boldsymbol{y} \rangle)_y$, where $\boldsymbol{y} \in \boldsymbol{S} \subseteq \{\boldsymbol{0,1}\}^n$

$$\langle \boldsymbol{x},\boldsymbol{y} \rangle = t_{1,1}(\boldsymbol{x},\boldsymbol{y}) = n - t_{0,1}(\boldsymbol{x},\boldsymbol{y}) - t_{1,0}(\boldsymbol{x},\boldsymbol{y}) - t_{0,0}(\boldsymbol{x},\boldsymbol{y})$$

$$x = (1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1)$$

$$y = (0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0)$$

Coin weighing problem

Equivalently, any $\boldsymbol{x} \in \{\boldsymbol{0,1}\}^n$ is uniquely determined by its **inner product vector** $(\langle \boldsymbol{x},\boldsymbol{y}\rangle)_{\boldsymbol{y}}$, where $\boldsymbol{y} \in \boldsymbol{S} \subseteq \{\boldsymbol{0,1}\}^n$

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle = \boldsymbol{t}_{1,1}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{n} - \underbrace{\boldsymbol{t}_{0,1}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{t}_{1,0}(\boldsymbol{x},\boldsymbol{y})}_{\boldsymbol{d}(\boldsymbol{x},\boldsymbol{y})} - \boldsymbol{t}_{0,0}(\boldsymbol{x},\boldsymbol{y})$$

Coin weighing problem

Equivalently, any $\boldsymbol{x} \in \{\boldsymbol{0},\boldsymbol{1}\}^n$ is uniquely determined by its **inner product vector** $(\langle \boldsymbol{x},\boldsymbol{y} \rangle)_{\boldsymbol{y}}$, where $\boldsymbol{y} \in \boldsymbol{S} \subseteq \{\boldsymbol{0},\boldsymbol{1}\}^n$

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Coin weighing problem

Equivalently, any $\boldsymbol{x} \in \{\boldsymbol{0,1}\}^n$ is uniquely determined by its **inner product vector** $(\langle \boldsymbol{x},\boldsymbol{y}\rangle)_{\boldsymbol{y}}$, where $\boldsymbol{y} \in \boldsymbol{S} \subseteq \{\boldsymbol{0,1}\}^n$

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle = \boldsymbol{t}_{1,1}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{n} - \underline{\boldsymbol{t}_{0,1}(\boldsymbol{x},\boldsymbol{y})} - \boldsymbol{t}_{1,0}(\boldsymbol{x},\boldsymbol{y}) - \underline{\boldsymbol{t}_{0,0}(\boldsymbol{x},\boldsymbol{y})}$$

of ***0***'s in ***x***

Coin weighing problem

Equivalently, any $\boldsymbol{x} \in \{\boldsymbol{0,1}\}^{\boldsymbol{n}}$ is uniquely determined by its **inner product vector** $(<\boldsymbol{x},\boldsymbol{y}>)_{\boldsymbol{y}}$, where $\boldsymbol{y} \in \boldsymbol{S} \subseteq \{\boldsymbol{0,1}\}^{\boldsymbol{n}}$

$$<\boldsymbol{x},\boldsymbol{y}> = \boldsymbol{n} - \boldsymbol{0.5}\Big(\boldsymbol{d}(\boldsymbol{x},\boldsymbol{y}) + \underbrace{\#\text{ of }\boldsymbol{0}\text{'s in }\boldsymbol{x}}_{\text{Cost: one weighing}} + \#\text{ of }\boldsymbol{0}\text{'s in }\boldsymbol{y}\Big)$$

Cost: one weighing

Coin weighing problem

Equivalently, any $\boldsymbol{x} \in \{\boldsymbol{0,1}\}^{n}$ is uniquely determined by its **inner product vector** $(<\boldsymbol{x},\boldsymbol{y}>)_{\boldsymbol{y}}$, where $\boldsymbol{y} \in \boldsymbol{S} \subseteq \{\boldsymbol{0,1}\}^{n}$

$$<\boldsymbol{x},\boldsymbol{y}> \; = \boldsymbol{n} - \boldsymbol{0.5}\Big(\boldsymbol{d}(\boldsymbol{x,y}) + \boldsymbol{\#} \text{ of } \boldsymbol{0}\text{'s in } \boldsymbol{x} + \underbrace{\boldsymbol{\#} \text{ of } \boldsymbol{0}\text{'s in } \boldsymbol{y}}_{\text{We know it!}}\Big)$$

Coin weighing problem

Equivalently, any $\boldsymbol{x} \in \{\boldsymbol{0,1}\}^n$ is uniquely determined by its **inner product vector** $(<\boldsymbol{x},\boldsymbol{y}>)_{\boldsymbol{y}}$, where $\boldsymbol{y} \in \boldsymbol{S} \subseteq \{\boldsymbol{0,1}\}^n$

$$<\boldsymbol{x},\boldsymbol{y}> \;= \boldsymbol{n} - \boldsymbol{0.5}\Big(\boldsymbol{d}(\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{\#} \text{ of } \boldsymbol{0}\text{'s in } \boldsymbol{x} + \boldsymbol{\#} \text{ of } \boldsymbol{0}\text{'s in } \boldsymbol{y}\Big)$$

Conclusion:

$|\boldsymbol{cw}(\boldsymbol{n})-\boldsymbol{m}(\boldsymbol{K_2},\boldsymbol{n})| \leq \boldsymbol{1}$

Timeline

1963 Erdös, Rényi	$m(K_2, n) \leq 3 \cdot n/\log_2 n$ $m(K_2, n) \geq 2 \cdot n/\log_2 n$
1964 Lindström ***1966 Cantor, Mills***	$m(K_2, n) = 2 \cdot n/\log_2 n$
1974 Chvátal	$m(K_q, n) \leq 2(2+\log_q 2)n/\log_q n$
1999 Kabatianski	$m(K_q, n) \geq 2 \cdot n/\log_q n$ $m(K_q, n) = 2 \cdot n/\log_q n$ for $q=3,4$
2017 Jiang, P.	$m(G, n) = 2 \cdot n/\log_q n$ for ***complete graphs, paths, cycles, complete bipartite graphs***

Open problems

We prove that for any* $\boldsymbol{G}$ on at most $\boldsymbol{q} \le \boldsymbol{9}$ vertices

$$\boldsymbol{m}(\boldsymbol{G},\boldsymbol{n}) = \boldsymbol{2}\cdot\boldsymbol{n}/\log_{\boldsymbol{q}}(\boldsymbol{n})$$

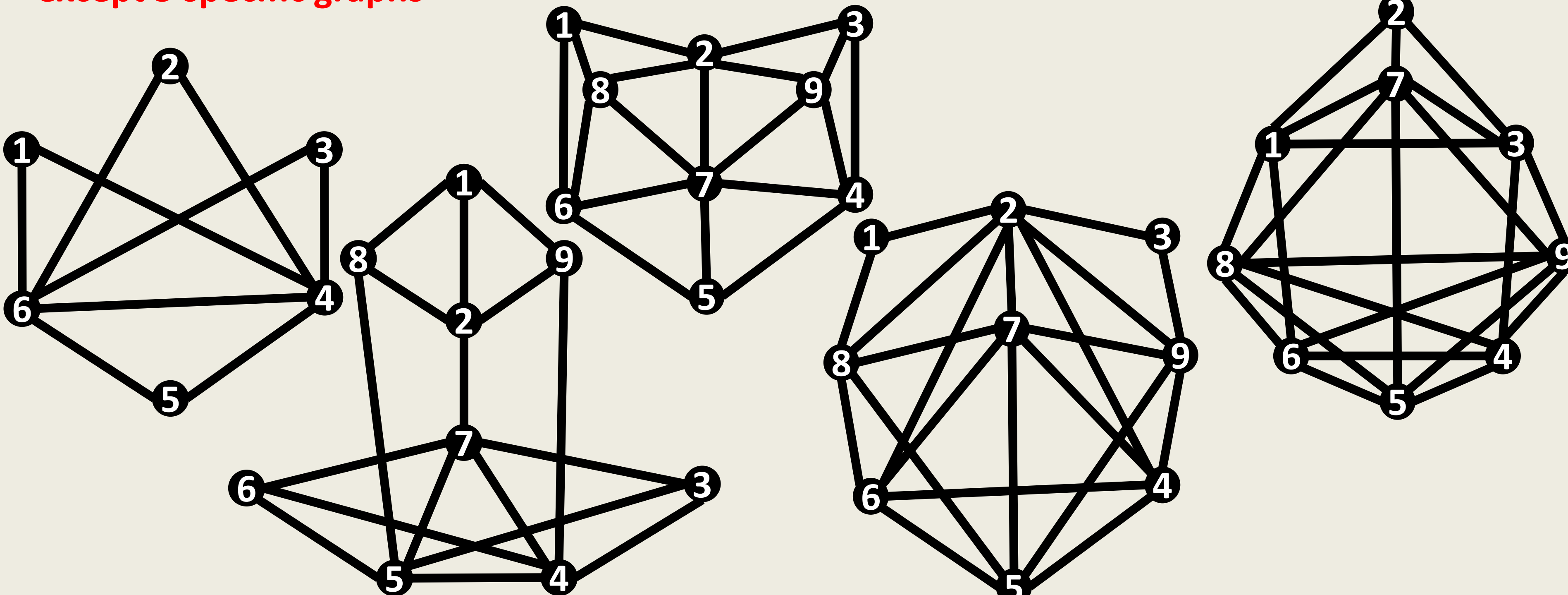
*** except 5 specific graphs**

Open problems

We prove that for any* G on at most $q \le 9$ vertices

$$m(G,n) = 2 \cdot n / \log_q(n)$$

* except 5 specific graphs



Open problems

Conjecture:

Given graph $\boldsymbol{G}$ on $\boldsymbol{q}$ vertices

$$\boldsymbol{m}(\boldsymbol{G},\boldsymbol{n}) = \boldsymbol{2}\cdot\boldsymbol{n}/\log_{\boldsymbol{q}}(\boldsymbol{n})$$

Thank you!