Uniform concentration results for the largest eigenvalue of the adjacency matrix in Erdős-Rényi processes

Nikita Zhivotovskiy[1]

[1]Technion, IITP

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Problem statement

We are given

- n vertices.
- For each pair of vertices i, j, $i \neq j$ we generate an independent $U_{ij} \sim U[0, 1]$.
- Vertices i, j are connected iff $U_{ij} \leq p$.

A_p denotes an adjacency matrix.

When p is fixed — $G(n, p)$ model (Erdős-Rényi model).

Denote the largest eigenvalue of A_p by λ_p. In our case $\lambda_p = \|A_p\|$.

Question

Understand the behaviour of $\sup_{p \in [0,1]} (\lambda_p - \mathbb{E}\,\lambda_p)$

Motivation

- Hypothesis testing (for random graphs).
- Analysis of the spectrum of random matrices with nonzero mean.
- Empirical process theory understands very well

$$\sup_{g \in G} \left(\sum_{i=1}^{n} g(X_i) - n\,\mathbb{E}\,g(X) \right),$$

where $X_1, \ldots, X_n$ are independent r.v. . Contrary to

$$\sup_{p \in [0,1]} (\lambda_p - \mathbb{E}\,\lambda_p),$$

where λ_p is a nonlinear function of independent arguments.

What is known for a fixed value of p?

1. By Efron-Stein for all n and $p \in [0,1]$ we have $\mathbf{Var}(\lambda_p) \leq 16$.
2. Matrix concentration gives for all $p \in [0,1]$ and any $t \geq 0$

$$P(\lambda_p - \mathbb{E}\,\lambda_p \geq t) \leq \exp\left(-t^2/32\right).$$

   by (Alon, Krivilevich, Vu' 2001).
3. Asymptotic behaviour when $p > 0$ is an absolute constant is

$$\lambda_p - \mu \xrightarrow{d} \mathcal{N}(0, 2p(1-p)),$$

   where $\mu = (n-1)p + (1-p)$ by (Fűredi and Komlos' 1981)
4. Denote $\Delta =$ max. degree. As $\max(np, \sqrt{\Delta}) \to \infty$ it holds with probability tending to 1

$$\lambda_p = (1 + o(1)) \max(np, \sqrt{\Delta})$$

   by (Krivilevich, Sudakov' 2001)
5. For the centered matrix $\mathbb{E}\,\|A_p - \mathbb{E}\,A_p\| \lesssim \sqrt{pn} + \sqrt{\log n}$. (Bandeira, Van Handel' 2016).

We prove the following uniform result.

Theorem

For any n and $t \geq c_1$

$$P\left(\sup_{p \in [\frac{c_2 \log(n)}{n}, 1]} (\lambda_p - \mathbb{E}\,\lambda_p) \geq t\right) \geq c_3 \exp(-c_4 t^2).$$

Question

Why $p \gtrsim \frac{\log n}{n}$?

Several words about the *sparse* regime ($p < \frac{\log n}{n}$).

- With high probability the graph corresponding to A_p will be disconnected.
- Graph will have isolated vertexes with a high degree.
- We have $\|A_p\| \gg \|\mathbb{E} A_p\|$. No concentration!
- Nothing very general we can say about eigenvectors. No delocalization!
- The behaviour is very different from the matrix $\|W_p\|$ with Bernoulli replaced with independent $\mathcal{N}(p, p(1-p))$.

We discuss it below.

The proof of the main theorem consists of several parts.

Lemma: Uniform high probability upper bound

For $q \in [\frac{\log n}{n}, \frac{1}{2}]$ *it holds with probability at least* $1-\exp(-c_1 nq)$

$$\sup_{p\in[q,2q]} \|A_p - \mathbb{E}\, A_p\| \le c_2\sqrt{nq}.$$

Observe that

$$\|A_p\| - \mathbb{E}\,\|A_p\| \le \|A_p - \mathbb{E}\, A_p\|.$$

A typical deviation of $\|A_p\| - \mathbb{E}\,\|A_p\|$ is of order $O(1)$,

while for $\|A_p - \mathbb{E}\, A_p\|$ it is equal to $\sqrt{np}$ when $p \ge \frac{\log n}{n}$.

This uniform result is one of the components to prove the following delocalization lemma.

Lemma: Uniform delocalization in $\| \|_2$

Given $q \in [\frac{c_1 \log(n)}{n}, \frac{1}{2}]$ *it holds with probability at least* $1 - c_2 \exp(-c_3 nq)$

$$\sup_{p \in [q, 2q]} \left\| v_p - \frac{\bar{1}}{\sqrt{n}} \right\|_2 \leq \frac{c_4}{\sqrt{nq}},$$

where v_p *is a unit eigenvector corresponding to* λ_p *and* $\frac{\bar{1}}{\sqrt{n}} = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$.

There is no delocalization when $p \lesssim \frac{\log n}{n}$.

Lemma: Uniform control over the expectation

For all n it holds

$$\mathbb{E} \sup_{p \in [\frac{c_1 \log(n)}{n}, 1]} (\lambda_p - \mathbb{E}\,\lambda_p) \leq c_2.$$

Proof idea: when $v_p \approx \frac{\bar{1}}{\sqrt{n}}$ we have

$$\lambda_p = \|A_p\| = v_p^T A_p v_p \approx \frac{\bar{1}}{\sqrt{n}}^T A_p \frac{\bar{1}}{\sqrt{n}} \sim \frac{2}{n} Bin\left(\binom{n}{2}, p\right).$$

Denoting $Y_p = \frac{\bar{1}}{\sqrt{n}}^T A_p \frac{\bar{1}}{\sqrt{n}}$ we use the Dvoretzky–Kiefer–Wolfowitz inequality.

Lemma: DKW inequality

$$P(\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \geq t) \leq 2\exp(-2nt^2).$$

Applying this to Y_p and integrating the tail we obtain

$$\mathbb{E} \sup_{p \in [\frac{c_1 \log(n)}{n}, 1]} |Y_p - \mathbb{E}\, Y_p| \leq C.$$

and the lemma follows. This is the place where the chaining techniques are hidden.

Finally, denoting $Z = \sup_{p \in [\frac{c_2 \log(n)}{n}, 1]} (\lambda_p - \mathbb{E}\,\lambda_p)$ we prove (using tensorization methods).

$$P\left(Z - \mathbb{E}\,Z \geq t\right) \lesssim c_2 \exp(-c_3 t^2).$$

Using $\mathbb{E}\,Z \leq c_4$ we finish the proof of our main theorem.

Asymptotic behaviour when p is an absolute constant.

$$\lambda_p - \mu \xrightarrow{d} \mathcal{N}(0, 2p(1-p)),$$

where $\mu = (n-1)p + (1-p)$ (by Fűredi and Komlos' 1981).

Question

Is it possible to obtain a similar behaviour in the non-asymptotic framework for various values of p?

Reminder: standard non-asymptotic results only give $\mathbf{Var}(\lambda_p) \leq 16$ independent of p.

Lemma: Delocalization in $\| \|_\infty$

For any n and $p \in [\frac{c_1 \log^3(n)}{n}, 1]$ *it holds with probability at least* $1 - c_2 n \exp\left(-c_3 \left(\frac{\log(np)}{\log(n)}\right)^2 np\right)$

$$\left\|v_p\right\|_\infty \leq \frac{c_4}{\sqrt{n}}.$$

Notice:

1. p is fixed (a nonuniform result).
2. No log-factors in $\| \|_\infty$ bound.
3. Exponential tail guaranties.

Theorem

For any n and $p \in [\frac{c_1 \log^3(n)}{n}, 1]$ and all k such that $2 \leq k \leq \frac{c_2\left(\frac{\log(np)}{\log(n)}\right)^2 pn}{\log(\frac{1}{p})}$ it holds

$$\mathbb{E}(\lambda_p - \mathbb{E}\,\lambda_p)_+^k \leq (c_3 kp)^{\frac{k}{2}}.$$

In particular, if p is going to zero not too slow we have the Gaussian concentration as $n \to \infty$ with variance parameter proportional to p.

Strategy of the proof: Tensorization of the variance (higher moments) + $\| \ \|_\infty$-delocalization:

$$\mathbf{Var}(f(X_1, \dots, X_n)) \leq \mathbb{E}\left(\sum_{i=1}^{n} \mathbf{Var}_i(f(X_1, \dots, X_n))\right),$$

where $\mathbf{Var}_i$ is a variance with respect to the i-th r.v. given the other.

Using the moment bound with $k = t^2/(2Cp)$ and Markov's inequality the last result implies that for all $0 < t \le \frac{cp\sqrt{n}\log(np)}{\log n \log(1/p)}$,

$$P\left(\|A_p\| > \mathbb{E}\,\|A_p\| + t\right) \le \exp\left(-\frac{t^2}{Cp}\right) \ .$$

which improves the bound of Alon, Krivilevich, Vu.

$$P(\|A_p\| - \mathbb{E}\,\|A_p\| \ge t) \le \exp\left(-\frac{t^2}{32}\right)$$

in this range.

Several words about the regime ($p \ll \frac{\log n}{n}$).
It follows that

$$\mathrm{Var}(\|A_p\|) = O(p).$$

for $p \geq \frac{\log^3 n}{n}$.

Question

Maybe $\mathrm{Var}(\|A_p\|) = O(p)$ *for all* $p \in [0, 1]$*?*

No! Take $p = c/n^2$ for a positive constant c.

- The probability that the graph $G(n, p)$ is empty is bounded away from zero. In that case $\|A_p\| = 0$.
- With a probability bounded away from zero, the graph $G(n, p)$ contains a single edge, in which case $\|A_p\| = 1$. Thus, $\mathrm{Var}(\|A_p\|) = \Omega(1)$.

Several words about the regime ($p \ll \frac{\log n}{n}$).
We showed above that

$$\mathbb{E} \sup_{p \in [\frac{c_1 \log(n)}{n}, 1]} (\lambda_p - \mathbb{E}\,\lambda_p) \leq c_2.$$

But the following bound holds

Lemma: Uniform control over the expectation

For all n it holds

$$\mathbb{E} \sup_{p \in [0,1]} (\lambda_p - \mathbb{E}\,\lambda_p) \leq c_2 \sqrt{\log \log n}.$$

Observe that for $p = \frac{1}{2}$ we have $\lambda_p \sim \sqrt{n}$.

That's all.