Algebraic Geometry over Abelian Groups

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Universal algebraic geometry

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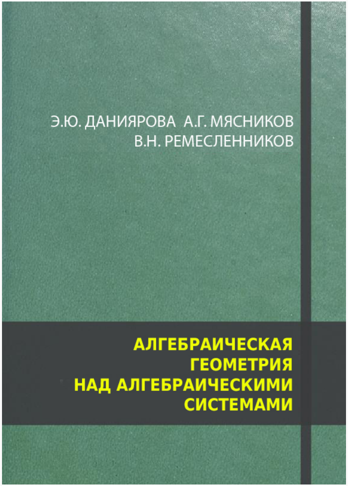
Algebraic geometry over algebraic structures

in an arbitrary language $\mathtt{L}$

G. Baumslag, A. Myasnikov, V. Remeslennikov
Algebraic geometry over groups I: Algebraic sets and ideal theory
J. Algebra, 219, 1999, 16–79

A. Myasnikov, V. Remeslennikov
Algebraic geometry over groups II: Logical foundations
J. Algebra, 234, 2000, 225–276

B. Plotkin
Seven lectures on the universal algebraic geomtry
arXiv, 2002, 87



E. Yu. Daniyarova, A. G. Myasnikov, V. N. Remeslennikov, *Algebraic geometry over algebraic structures*, Novosibirsk: Publ. SB RAS, 2016, 243 p.

Transfer

Universal algebraic geometry

Classical algebraic geometry over a field

Main definitions + New definitions

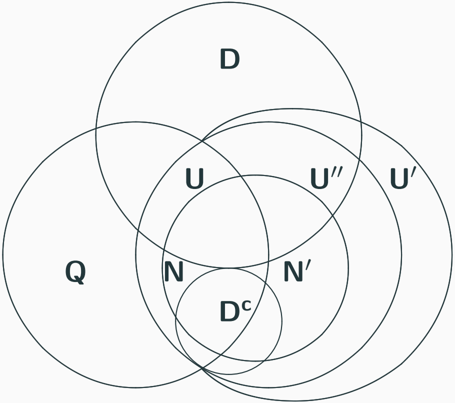
Main problems + New problems

Main results + New results

true for all algebraic structures

form a ``true set'' of algebraic structures

The map: Special classes of algebraic structures = "True sets"



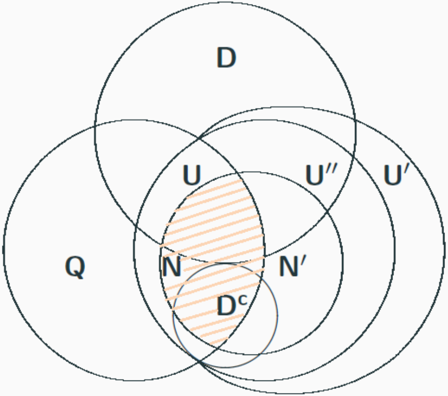
This is how the universal algebraic geometry works in practice

Take a language $\mathtt{L} = \{\text{constants, functions, relations}\}$ and an algebraic structure $\mathcal{A}$ in $\mathtt{L}$.

For example, when studying

- simple graphs we take $\mathtt{L} = \{E(x, y)\}$,
- orders — $\mathtt{L} = \{\leqslant\}$,
- lattices — $\mathtt{L} = \{\vee, \wedge, \leqslant\}$,
- abelian groups — $\mathtt{L} = \{+, -, 0, \text{other constants}\}$, and so on.

Abelian groups



Theorem

All abelian groups are equationally Noetherian.

Main definitions

Let A be an abelian group.

- Equations over A in the variables $x_1, \ldots, x_n$ have the form

$$m_1 x_1 + \ldots + m_n x_n = a, \quad m_i \in \mathbb{Z}, a \in A.$$

- For a system of equations S the set $\mathrm{V}_A(S) \subseteq A^n$ of all its solutions in A is called an algebraic set over A.
- All algebraic sets $Y = \mathrm{V}_A(S)$ are subdivided into reducible ($Y = Y_1 \cup \ldots \cup Y_m$), irreducible ($Y \neq Y_1 \cup \ldots \cup Y_m$) and the empty set $\emptyset$.
- For a system of equations S the maximal system of equations $\mathrm{Rad}_A(S)$ that is equivalent to S (i. e., has the same set of solutions as S) is called the radical of S.
- The quotient-group $\mathbb{Z}^n \oplus A/\mathrm{Rad}_A(S)$ is called the coordinate groups of the algebraic set $Y = \mathrm{V}_A(S)$ and denoted by $\Gamma(Y)$.

The main problems of algebraic geometry over A:

1. To classify algebraic sets over A with accuracy up to isomorphism;
2. To classify irreducible algebraic sets over A with accuracy up to isomorphism;
3. To classify coordinate groups of algebraic sets over A;
4. To classify coordinate groups of irreducible algebraic sets over A.

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4. To classify coordinate groups of irreducible algebraic sets over A.

Theorem

Every non-empty algebraic set Y over A can be expressed as a finite union of irreducible algebraic sets (irreducible components):

$$Y = Y_1 \cup \ldots \cup Y_m.$$

Furthermore, this decomposition is unique up to permutation of irreducible components and omission of superfluous ones.

The main problems of algebraic geometry over A:

1. To classify algebraic sets over A with accuracy up to isomorphism;
2. To classify irreducible algebraic sets over A with accuracy up to isomorphism;
3. To classify coordinate groups of algebraic sets over A;
4. To classify coordinate groups of irreducible algebraic sets over A.

Theorem

The category of algebraic sets over A and the category of their coordinate groups are dually equivalent.

The main problems of algebraic geometry over A:

1. To classify algebraic sets over A with accuracy up to isomorphism;
2. To classify irreducible algebraic sets over A with accuracy up to isomorphism;
3. To classify coordinate groups of algebraic sets over A;
4. To classify coordinate groups of irreducible algebraic sets over A;
5. To classify all abelian groups up to geometrical equivalence (by B. Plotkin);
6. To classify all abelian groups up to universal geometrical equivalence;
7. To classify abelian groups from special classes.

The elementary invariants of abelian groups

For an abelian group A, in 1955 Wanda Szmielew defined elementary invariants

$$\alpha_{p,k}(A),\ \beta_{p,k}(A),\ \gamma_{p,k}(A) \in \mathbb{N} \cup \{\infty\},\ \ \delta(A) \in \{0,1\},$$
$$k \in \mathbb{Z}^+, p \in \{\text{primes}\},$$

and proved that for abelian groups A_1 and A_2 one has

$$A_1 \equiv A_2 \quad \Longleftrightarrow \quad A_1 \text{ and } A_2 \text{ have the same elementary invariants.}$$

For a prime p the exponent is $e_p(A) = \sup\{k, \gamma_{p,k}(A) \neq 0\}$.

The solution of the main problems

Theorem (the classification of coordinate groups)

An abelian group C is the coordinate group of an algebraic set over A iff for some finitely generated group G one has

$$C \cong A \oplus G, \quad \delta(C) = \delta(A),\ e_p(C) = e_p(A) \text{ for every prime } p.$$

Theorem (the classification of irreducible coordinate groups)

An abelian group C is the coordinate group of an algebraic set over A iff for some finitely generated group G one has

$$C \cong A \oplus G, \quad \delta(C) = \delta(A),\ \gamma_{p,k}(C) = \gamma_{p,k}(A)$$
$$\text{for every prime } p \text{ and } k \in \mathbb{Z}^+.$$

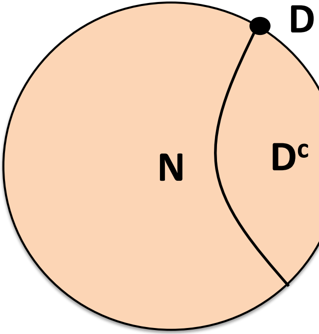
The description of algebraic sets

- Let $C \cong A \oplus G$ be the coordinate algebra of an algebraic set Y and $G \cong C(a_1) \oplus C(a_n) \oplus C(b_1) \oplus C(b_m)$, where $a_1, \dots, a_n$ have the infinite orders and $b_1, \dots, b_m$ have finite orders $k_1, \dots, k_m$, then

$$Y \cong (\underbrace{A, \dots, A}_{n}, A[k_1], \dots, A[k_m]).$$

- If Y is reducible, then it is easy to fined irreducible $iY \subset Y$, such that $\Gamma(iY) \subseteq \Gamma(Y)$. It tunes out that all irreducible components of Y are isomorphic to iY, and their quantity equals to $|\Gamma(Y) : \Gamma(iY)|$.

The map for abelian groups



N = {equationally Noetherian groups}

D = {equational domains} = {0}

$\mathbf{D^c}$ **=** {equational co-domains}

Definition

An abelian group A is called an equational co-domain, if all non-empty algebraic sets over A are irreducible.

For instance, all torsion-free abelian groups are equational co-domains.

Theorem

An abelian group A is an equational co-domain iff $\gamma_{p,k}(A) \in \{0, \infty\}$ for all prime p and $k \in \mathbb{Z}^+$.

Definition

Abelian groups A_1 and A_2 are called geometrically equivalent, if for any system of equations S one has

$$\mathrm{Rad}_{A_1}(S) = \mathrm{Rad}_{A_2}(S).$$

If additionally one has

$$\mathrm{V}_{A_1}(S) \text{ is irreducible } \iff \mathrm{V}_{A_2}(S) \text{ is irreducible },$$

then A_1 and A_2 are called universally geometrically equivalent.

Geometrical equivalences

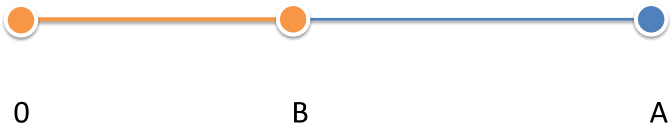
Theorem

Abelian groups A_1 and A_2 are geometrically equivalent iff $\delta(A_1) = \delta(A_2)$ and $e_p(A_1) = e_p(A_2)$ for all primes p.

Theorem

Abelian groups A_1 and A_2 are universally geometrically equivalent iff $\delta(A_1) = \delta(A_2)$ and $\gamma_{p,k}(A_1) = \gamma_{p,k}(A_2)$ for all primes p and $k \in \mathbb{Z}^+$.

Settings



subgroup of coefficients

- The coefficient-free case: $B = 0$
- The Diophantine case: $B = A$

As usual B is a pure subgroup in A.

The solution of the main problems in general case

Let B be a pure subgroup of an abelian group A.

Theorem (the classification of coordinate groups)

An abelian group C is the coordinate group of an algebraic set over A for a system of equations with coefficients in B iff for some f. g. group G one has $C \cong B \oplus G$ and

$$\delta(G) \leqslant \delta(A),\ e_p(G) \leqslant e_p(A) \text{ for every prime } p.$$

Theorem (the classification of irreducible coordinate groups)

An abelian group C is the coordinate group of an irreducible algebraic set over A for a system of equations with coefficients in B iff for some f. g. group G one has $C \cong B \oplus G$ and

$$\delta(G) \leqslant \delta(A),\ \gamma_{p,k}(G) \leqslant \gamma_{p,k}(A) \text{ for every prime } p \text{ and } k \in \mathbb{Z}^{+}.$$

The classification of equational co-domains in general case

Theorem

An abelian group A is an equational co-domain in the Diophantine case iff it is an equational co-domain with coefficients in B (with any choice of subgroup $B \leqslant A$).

Geometrical equivalences in general case

Theorem

Abelian groups A_1 and A_2 are geometrically equivalent in coefficient-free case iff they are geometrically equivalent with coefficients in a pure subgroup B of both A_1 and A_2 (with any choice of B).

Theorem

Abelian groups A_1 and A_2 are universally geometrically equivalent in coefficient-free case iff they are universally geometrically equivalent with coefficients in any pure subgroup B of both A_1 and A_2.

Thank you!