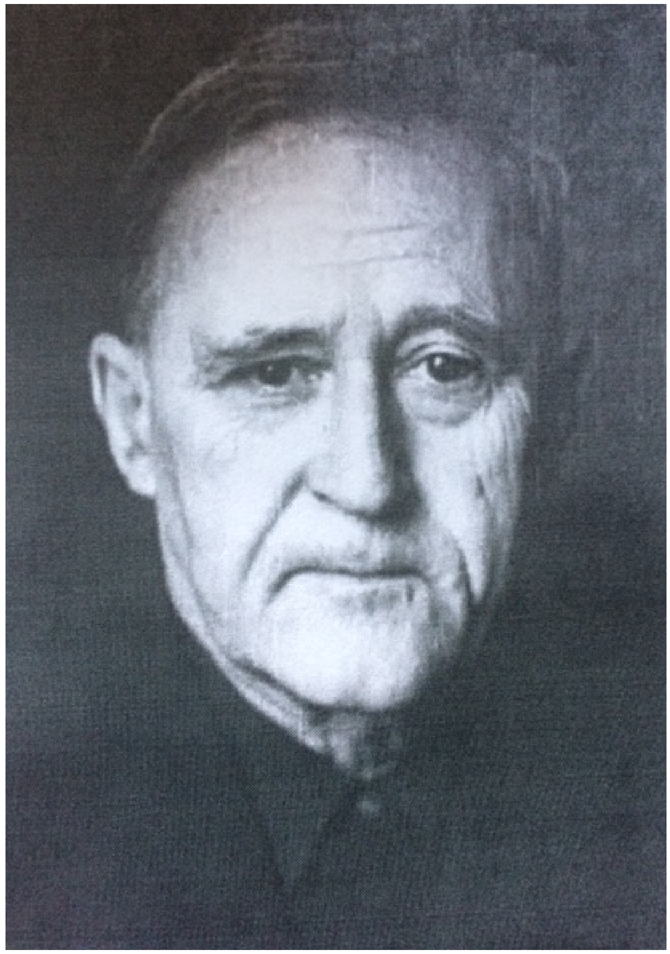
Transcendence and Diophantine Problems

Conference in memory of Professor Naum Ilyitch Feldman (1918 - 1994)

Moscow, June 10 - June 14, 2019

Program and Abstract Book



Professor Naum Ilyitch Feldman
26.11.1918 – 20.04.1994

List of Participants

Maria Bashmakova mariya-bashmakova@yandex.ru
Attila Bérczes berczesa@science.unideb.hu
Viktor Bykovskii vab@iam.khv.ru
Vladimir Chirskii vgchirskii@yandex.ru
Artūras Dubickas arturas.dubickas@mif.vu.lt
Alexander Galochkin aigalochkin@yandex.ru
Dmitry Gayfulin gamak.57.msk@gmail.com
Oleg German german.oleg@gmail.com
Anastasia Godunova icq13@mail.ru
Vasilii Gorelov gorelov.va@mail.ru
Aanchal Gupta gupta.aanchal@phystech.edu
Kalman Győry gyory@science.unideb.hu
Lajos Hajdu hajdul@science.unideb.hu
Noriko Hirata-Kohno hirata@math.cst.nihon-u.ac.jp
Pavel Ivankov ivankovpl@mail.ru
Sergei Konyagin konyagin23@gmail.com
Vladimir Lysov vlysov@mail.ru
Raffaele Marcovecchio raffaele.marcovecchio@unich.it
Evgenii Matveev matemix@mail.ru
Boris Moroz moroz.bb@mipt.ru
Nikolay Moshchevitin moshchevitin@gmail.com
Yuri Nesterenko nester@mi.ras.ru
Anton Popov ...
Arsenii Sagdeev xp1@protonmail.com
Vladislav Salikhov svdh@rambler.ru
Ilya Shkredov ilya.shkredov@gmail.com
Nikita Shulga nikita1279@yandex.ru
Vladimir Sorokin vladimirs1957@gmail.com
Sergei Stepanov sa-stepanov@iitp.ru
Carlo Viola viola@dm.unipi.it
Paul Voutier paul.voutier@gmail.com
Michel Waldschmidt waldschmidtmichel@gmail.com
Daria Zabrodina zabr.daria@gmail.com
Wadim Zudilin wzudilin@gmail.com

SCHEDULE

Monday 10.06.2019

Lectures will be held in Large Chemical Hall, MIPT (PHYSTECH)

09:30 - 10:10	**REGISTRATION**	at the Entrance of Laboratorny Building
10:10 - 10.50	Yuri Nesterenko	N.I. Feldman and transcendental numbers
11:00 - 11:50	Kalman Győry	Effective results for diophantine equations over finitely generated domains
12:00 - 12:50	Attila Bérczes	Some Diophantine problems connected to binary recurrences
	Lunch break	
15:00 - 15:40	Paul Voutier	Sharp bounds on the number of solutions of $X^2 - (a^2+b)\,Y^4 = -b$
15:50 - 16:20	Arsenii Sagdeev	On asymmetric Diophantine approximation

Last lecture will be held at room KPM 302

16:40 - 17:30	Viktor Bykovskii	On lengths of periods of quadratic irrationalities in average.

Tuesday 11.06.2019

Seminar of the Department of Number Theory,
Moscow Lomonosov State University, Main Building, room 13-06

For the participants of the conference a bus goes from the hotel at 09:00

11:00 - 11:50	Michel Waldschmidt	Integer valued entire functions and Lidstone series
12:00 - 12.30	Ilya Shkredov	Some progress in the problem of Zaremba
12:40 - 13:20	Pavel Ivankov	On Linear Independence of certain functions
	Lunch break	
15:00 - 15:50	Lajos Hajdu	Multiplicative decompositions of polynomial sequences
16:00 - 16:40	Vasilii Gorelov	On algebraic identities between solution matrices of generalized hypergeometric equations
16:50 - 17:10	Daria Zabrodina	On an identity for hypergeometric type integrals
17:20 - 17:40	Vladimir Chirskii	Arithmetic properties of certain sereis
17:50 - 18:20	Boris Moroz	Diophantine equations (variations on the themes of Yu.V. Matiyasevich)
18:30 - 18:50	Nikolay Moshchevitin	A few words about Naum Ilyitch Feldman

Wednesday 12.06.2019

Lectures will be held in room 110 KPM, MIPT (PHYSTECH)

10:00 - 10:40	Vladislav Salikhov	On estimates of simultaneous approximation to the rational numbers logarithms
10:50 - 11:20	Maria Bashmakova	On estimates of irrationality measures of certain values of $\arctan 1/n$
11:30 - 11:50	Anastasia Godunova, Alexander Galochkin	On approximations of solutions of the equation $P(z, \log z) = 0$ by algebraic numbers
12:00 - 12:20	Vladimir Lysov	On the irrationality and the irrationality exponent of the product of two logarithms
	Lunch break	
15:00 - 15:40	Carlo Viola	The permutation group method: results and problems
15:50 - 16:30	Raffaele Marcovecchio	Hypergeometric rational approximations to $\zeta(4)$
16:40 - 17:30	Wadim Zudilin	Many odd zeta values are irrational
17:40 - 18:20	Evgenii Matveev	On numbers of bounded height from a fixed algebraic field

Thursday 13.06.2019

Lecture will be held at room KPM 302, MIPT (PHYSTECH)

10:00 - 10:30	Dmitry Gayfulin	On the derivative of Minkowski question-mark function
10:40 - 11.10	Artūras Dubickas	Multiplicative dependence of translated algebraic numbers
11:20 - 12:00	Noriko Hirata-Kohno	Linear forms in logarithms: contribution by Fel'dman and developments
	Lunch break	
15:00 - 15:30	Nikita Shulga	Diophantine properties of fixed points of Minkowski question mark function
15:40 - 16:10	Oleg German	Generalization of Dyson's transference theorem to the weighted setting

Friday 14.06.2019

Free day for discussions

ABSTRACTS

Maria Bashmakova. *On estimates of the irrationality measure for some values of* $\arctan 1/n$.

The subject of the research is irrationality measure estimates for some values of $\arctan 1/n$, which are based on symmetrized complex integrals. We add to underlying integrals symmetric polynomials of special type, which improve arithmetic properties of the linear forms for approximation of these values. Application of this method allows to get new estimates for measure of irrationality of $\arctan 1/2, arctan 1/3$ which are better than previously known results.

Attila Bérczes. *Some Diophantine problems connected to binary recurrences.*

Binary recurrence sequences are in the focus of research for a long time. In the frame of Diophantine number theory researchers were investigating divisibility properties of members of recurrence sequences and many more Diophantine properties of them. However, there are several Diophantine problems and Diophantine equations which are not directly connected to recurrences, but linear recurrences (and theorems proved about them) appear as tools in the solution of these problems.

In my talk I will present some results on Diophantine equations and problems containing linear recurrence sequences, and also some diophantine results, where the linear recurrences appear only in the proof of the results.

Viktor Bykovskii. *On lengths of periods of quadratic irrationalities in average.*

Let $D > 1$ be a fundamental discriminant. The quadratic form

$$Q(X, Y) = aX^2 + bXY + cY^2$$

with integes a, b, c and the discriminant

$$b^2 - 4ac = D$$

is reduced iff

$$0 < b < \sqrt{D}, \qquad \sqrt{D} - b < 2|c| < \sqrt{D} + b.$$

Let

$$x_1 = \frac{b}{2c} + \frac{\sqrt{D}}{2c}, \qquad x_2 = \frac{b}{2c} - \frac{\sqrt{D}}{2c}$$

be roots of quadratic equation

$$a - bx + cx^2 = 0.$$

It is know that for reduced quadratic forms Q the continued fraction expansion of $|x_1|$ is purely periodic

$$|x_1| = [\overline{a_0; a_1, \ldots, a_{l-1}}]$$

with period $(a_0, a_1, \ldots, a_{l-1})$ of the length l. Let $l_1, \ldots, l_{h(D)}$ be the periods of all non-equivalent reduced forms with discriminant D.

Theorem. For some $\delta > 0$

$$\sum_{i=1}^{h(D)} l_i = \frac{6 \log 2}{\pi^2} \sqrt{D} L(1; \chi_D) + O\left(D^{\frac{1}{2} - \delta}\right),$$

$$L(s, \chi_D) = \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) \frac{1}{n^s},$$

where

$$\chi_D(n) = \left(\frac{D}{n}\right)$$

is the Kronecker symbol.

Previous result in this direction was conditional, see [1].

This research is supported by the Russian Science Foundation (project 19-11-33365).

References

[1] L A. Takhtadzhyan, *An asymptotic formula for the sum of the lengths of the periods of quadratic irrationalities with discriminant*, Journal of Soviet Mathematics, November 1981, Vol. 17, Issue 5, p. 2173–2181.

Vladimir Chirskii. *Arithmetic properties of some series.*

Recently, the interest of researchers (V. V. Zudilin, T. Matala-Aho, etc.) was attracted by the problem of arithmetic properties of the Euler series:

$$\mathbf{e} = \sum_{n=0}^{\infty} n!.$$

Before that, my works proved the infinite transcendence of this series. This means that for any non-identical zero polynomial $P(x)$ there exists an infinite set of primes p such that $P(\mathbf{e}^p) \neq 0$ in the field $\mathbb{Q}_p$, where $\mathbf{e}^p$ denotes the sum of the Euler series in this field. In the works of the above-mentioned authors, we obtain a refinement consisting in the fact that for the linear form $L(x)$, different from the identical zero, there is an infinite set of primes p such that $L(\mathbf{e}^p) \neq 0$ in the in the field $\mathbb{Q}_p$, and we consider not all the set of primes, but some of its own subsets. The report describes this and related problems.

Artūras Dubickas *On multiplicative dependence of the translations of algebraic numbers*

Given pairwise distinct algebraic numbers $\alpha_1, \ldots, \alpha_n$, we are interested whether the shifted numbers $\alpha_1 + t, \ldots, \alpha_n + t$ are multiplicatively dependent or independent for $t \in \mathbb{N}$. In particular, we show that the above numbers are multipicatively independent for each sufficiently large $t \in \mathbb{N}$.

Further, for a fixed pair of distinct integers (a, b) , we study how many pairs $(a + t, b + t)$ are multiplicatively dependent when t runs through the set of integers $\mathbb{Z}$. Assuming the ABC conjecture we show that there exists a constant C_1 such that for any pair $(a, b) \in \mathbb{Z}^2$, $a \neq b$, there are at most C_1 values of $t \in \mathbb{Z}$ such that $(a + t, b + t)$ are multiplicatively dependent. For a pair $(a, b) \in \mathbb{Z}^2$ with difference $b - a = 30$ we show that there are 13 values of $t \in \mathbb{Z}$ for which the pair $(a + t, b + t)$ is multiplicatively dependent. We further conjecture that 13 is the largest number of such translations for any such pair (a, b) and prove this for all pairs (a, b) with difference at most 10^{10}.

The results are joint with Min Sha (Sydney).

Dmitry Gayfulin. *Minkowski question-mark function: fixed points and the derivative*

This is a joint work with Igor Kan.

The Minkowski question-mark function $?(x)$ is a continuious and strictly increasing function, which maps the $[0, 1]$ interval onto itself. If $[0; a_1, a_2, \ldots, a_n, \ldots]$ is the continued fraction representation of an irrational number x, then

$$?(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+\ldots+a_k-1}}$$

If x is a rational number, then in the formula behind is replaced by a finite sum. It is well known fact that the derivative of $?(x)$ can take exactly two values - 0 and $+\infty$. I will tell about new results on sufficient and necessary conditions on x such that the derivative equals 0, or conversely $+\infty$.

Oleg German. *Generalization of Dyson's transference theorem to the weighted setting.*

The talk is devoted to generalizing Dyson's transference inequalities to the weighted setting.

Let us fix weights $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^m_{>0}$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n_{>0}$,

$$\sigma_1 \geqslant \ldots \geqslant \sigma_m, \qquad \rho_1 \geqslant \ldots \geqslant \rho_n, \qquad \sum_{j=1}^{m} \sigma_j = \sum_{i=1}^{n} \rho_i = 1.$$

Let us define the weighted norms $|\cdot|_{\boldsymbol{\sigma}}$ and $|\cdot|_{\boldsymbol{\rho}}$ by

$$|\mathbf{x}|_{\boldsymbol{\sigma}} = \max_{1\leqslant j\leqslant m} |x_j|^{1/\sigma_j} \qquad \text{for } \mathbf{x} = (x_1, \ldots, x_m),$$

$$|\mathbf{y}|_{\boldsymbol{\rho}} = \max_{1\leqslant i\leqslant n} |y_i|^{1/\rho_i} \qquad \text{for } \mathbf{y} = (y_1, \ldots, y_n).$$

Consider the system of inequalities

$$\begin{cases} |\mathbf{x}|_{\boldsymbol{\sigma}} \leqslant t \\ |\Theta\mathbf{x} - \mathbf{y}|_{\boldsymbol{\rho}} \leqslant t^{-\gamma} \end{cases} .$$

Definition. *The* weighted Diophantine exponent $\omega_{\boldsymbol{\sigma},\boldsymbol{\rho}}(\Theta)$ *is defined as the supremum of real γ such that the considered system of inequalities admits nonzero solutions in $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^{m+n}$ for some arbitrarily large t.*

We present the following theorem, which generalises Dyson's inequality to the weighted case. Let $\Theta^\top$ denote the transpose of Θ.

Theorem. *Set $\omega = \omega_{\boldsymbol{\sigma},\boldsymbol{\rho}}(\Theta)$ and $\omega^\top = \omega_{\boldsymbol{\rho},\boldsymbol{\sigma}}(\Theta^\top)$. Then*

$$\omega^\top \geqslant \frac{\left(\rho_n^{-1} - 1\right) + \sigma_m^{-1}\omega}{\rho_n^{-1} + \left(\sigma_m^{-1} - 1\right)\omega}.$$

Anastasia Godunova. *On approximations of solutions of the equation $P(z, \log z) = 0$ by algebraic numbers.*

This is joint work with Alexander Galochkin. Let $\zeta \in \mathbb{C}$ be a root of equation

$$P(z, \log z) = 0, \quad \text{where} \quad 0 \not\equiv P(x_1, x_2) \in \mathbb{Z}[x_1, x_2].$$

We prove that the inequality

$$|\zeta - \theta| < \exp\left(-\Lambda \frac{\log^2 h(\theta)}{\log\log h(\theta)}\right)$$

has not more than a finitely many solutions in algebraic numbers θ of bounded degree. The constant Λ depends only on the degree of the polynomial P. Here $h(\theta)$ is the height of the algebraic number θ.

Vasilii Gorelov. *On algebraic identities between solution matrices of generalized hypergeometric equations.*

Let $\mathbb{A}$ be the set of all algebraic numbers, $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^- = \mathbb{Z} \setminus \mathbb{N}$, δ_i^j be the Kronecker delta, $\mathbb{C}[z^{\pm 1}]$ be the ring $\mathbb{C}[z, z^{-1}]$, $GL(q, K)$ be the general linear group over arbitrary ring K.

The functions

$$ {}_l\varphi_q(z) = {}_l\varphi_q(\vec{\nu}; \vec{\lambda}; z) = {}_{l+1}F_q\left(\begin{matrix} 1, \nu_1, \ldots, \nu_l \\ \lambda_1, \ldots, \lambda_q \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(\nu_1)_n \ldots (\nu_l)_n}{(\lambda_1)_n \ldots (\lambda_q)_n} z^n, $$

where $0 \leqslant l \leqslant q$, $(\nu)_0 = 1$, $(\nu)_n = \nu(\nu+1)\ldots(\nu+n-1)$, $\vec{\nu} = (\nu_1, \ldots, \nu_l) \in \mathbb{C}^l$, $\vec{\lambda} \in (\mathbb{C} \setminus \mathbb{Z}^-)^q$ are called generalized hypergeometric functions (see [1, 2]).

The function ${}_l\varphi_q(\vec{\nu}; \vec{\lambda}; z)$ satisfies the (generalized) hypergeometric differential equation

$$ L(\vec{\nu}; \vec{\lambda}; z)\, y = (\lambda_1 - 1)\ldots(\lambda_q - 1), $$

where

$$ L(\vec{\nu}; \vec{\lambda}; z) \equiv \prod_{j=1}^{q} (\delta + \lambda_j - 1) - z \prod_{k=1}^{l} (\delta + \nu_k), \quad \delta = z\frac{d}{dz}. $$

If $\lambda_i - \lambda_k \notin \mathbb{Z}$, $i \neq k$, and $q \geqslant \max(1, l)$, then the functions

$$ z^{(1-\lambda_k)p}{}_lF_{q-1}(\vec{\nu} + 1 - \lambda_k; \vec{\lambda} + 1 - \lambda_k; \alpha z^p), \quad k = 1, \ldots, q \tag{1} $$

constitute the fundamental system of solution of equation $L(\vec{\nu}; \vec{\lambda}; \alpha z^p)y = 0$, obtained from $L(\vec{\nu}; \vec{\lambda}; z)y = 0$ by the substitution $z \to \alpha z^p$.

The Siegel-Shidlovskii method (see [1], [2]) permits to establish the transcendency and the algebraic independence of the values of entire functions of some class, which contains the functions ${}_l\varphi_q(\alpha z^{q-l})$, provided that these functions are algebraically independent over $\mathbb{C}(z)$.

Let Φ_1, Φ_2 be arbitrary solution matrices of two linear homogeneous differential equations. If $\Phi_1 = gB\Phi_2 C$ or $\Phi_1(\Phi_2 C)^T = gB$, where $C \in GL(q, \mathbb{C})$, $B \in GL(q, \mathbb{C}(z))$, $g'/g \in \mathbb{C}(z)$, then these differential equations are said to be cogredient or contragredient respectively (see [3]). These notions are important for determination of algebraic dependence and independence of functions.

For the vectors $\vec{\mu} = (\mu_1, \ldots, \mu_n)$, $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ we shall write $\vec{\mu} \sim \vec{\eta}$, if there exists a permutation π of the numbers $1, \ldots, n$ such that $\mu_i - \eta_{\pi(i)} \in \mathbb{Z}$, $i = 1, \ldots, n$. We shall also use the notation $\gamma\vec{\mu} + \beta = (\gamma\mu_1 + \beta, \ldots, \gamma\mu_n + \beta)$, where $\gamma, \beta \in \mathbb{C}$.

Theorem 1. *Suppose that* $\vec{\nu} \in \mathbb{C}^l$, $\vec{\lambda} \in \mathbb{C}^q$, $q \geqslant \max(2, l)$, $\alpha \in \mathbb{C}$, $p \in \mathbb{N}$, *and* Φ_1, Φ_2 *are arbitrary solution matrices of the differential operators* $L(\vec{\nu}; \vec{\lambda}; \alpha z^p)$ *and* $L(1 - \vec{\nu}; 2 - \vec{\lambda}; (-1)^{q-l}\alpha z^p)$. *Then*

1° *there exists a matrix* $C \in GL(q, \mathbb{C})$ *such that*

$$ \Phi_1(\Phi_2 C)^T = B, \tag{2} $$

where $B = \|b_{i,j}\|_{i,j} \in GL(q, \mathbb{C}[z^{\pm 1}, (1 - \alpha z^p)^{\varepsilon}])$, $\varepsilon = -\delta_q^l$, *with*

$$ b_{k,q-k+1} = (-1)^k c_0 z^{1-q}(1 - \alpha z^p)^{\varepsilon}, \quad c_0 \in \mathbb{C}, \quad k = 1, \ldots, q, $$

and all the other entries above these entries are zeros;

2° *if* $\lambda_i - \lambda_k \notin \mathbb{N}$, $i, k = 1, \ldots, q$, *while* Φ_1, Φ_2 *correspond to the sets of functions (1) and*

$$ f_k = z^{(\lambda_k - 1)p}{}_lF_{q-1}(\lambda_k - \vec{\nu}; \lambda_k + 1 - \vec{\lambda}; (-1)^{q-l}\alpha z^p), \; k = 1, \ldots, q, $$

then in equation (2) $C = \mathrm{diag}(c_1, \ldots, c_q)$,

$$c_k = (-1)^k \prod_{1 \leqslant i < j \leqslant q;\ i,j \neq k} (\lambda_i - \lambda_j); \qquad c_0 = p^{q-1} \prod_{1 \leqslant i < j \leqslant q} (\lambda_i - \lambda_j);$$

the empty product of the brackets is equal to 1.

Theorem 2. *Suppose that* $\vec{\nu} \in \mathbb{C}^l$, $\vec{\lambda} \in \mathbb{C}^q$, $q \geqslant \max(2, l)$, $\alpha \in \mathbb{C}$, $p \in \mathbb{N}$, *and* Φ_1, Φ_2 *are arbitrary solution matrices of the differential operators* $L(\vec{\nu}; \vec{\lambda}; \alpha z^p)$ *and* $L(\vec{\nu} + 1 - \lambda_s; \vec{\lambda} + 1 - \lambda_s; \alpha z^p)$, $1 \leqslant s \leqslant q$. *Then*

$$\Phi_1 = z^{(1-\lambda_s)p} B \Phi_2 C,$$

where B *is a lower-triangular matrix from* $SL(q, \mathbb{C}[z^{-1}])$ *with 1' in the main diagonal and* $C \in GL(q, \mathbb{C})$.

Theorems 1–2 give nontrivial examples of cogredience and contragredience.

Functions, whose parameters differ from those of the original hypergeometric function φ by integer values are said to be associated with φ. Differential equations, that correspond to associated functions, are cogredient (see [4, lemma 12], [5]).

Next theorem is an example of cogredience and contragredience in the case of not coincident values of l.

Theorem 3. *The equations* $L(\vec{\nu}_k; \vec{\lambda}_k; \alpha_k z^{p_k})y = 0$, *where* $\alpha_k \in \mathbb{C}$, $p_k \in \mathbb{N}$, $\vec{\nu}_k = (\nu_{k,1}, \ldots, \nu_{k,l_k}) \in \mathbb{C}^{l_k}$, $\vec{\lambda}_k = (\lambda_{k,1}, \ldots, \lambda_{k,q_k}) \in (\mathbb{C} \setminus \mathbb{Z}^-)^{q_k}$, $k = 1, 2$, *with the condition*

$$q_1 = q_2 = 2,\ l_1 = 1,\ l_2 = 0,\ p_2 = 2p_1,\ \alpha_1^2 = 16\alpha_2,$$

$$\vec{\lambda}_1 - \lambda_{1,j} \sim \pm 2(\vec{\lambda}_2 - \lambda_{2,1}),\ 1 \leqslant j \leqslant 2,\ 2\nu_{1,1} - \lambda_{1,1} - \lambda_{1,2} \in \mathbb{Z},$$

are cogredient and contragredient.

Theorem 3 corrects some errors in [3] (see [6] for details).

According to results of [4], theorems 1–3 and [5] reveals all the causes of cogredience and contragredience of hypergeometric equations, provided that the entries of each solution matrix are bound only by Liouville relation.

Reference.

1. *Siegel C.L.* Über einige Anwendungen Diophantischer Approximationen // Abh. Preuss. Acad. Wiss., Phys. – Math. Kl. – 1929 – 1930. – no. 1. – S. 1 – 70.

2. *Shidlovskii A. B.* Transcendental numbers. – Moscow: Nauka, 1987.

3. *Beukers F., Brownawell W.D., Heckman G.* Siegel normality // Annals of Math. – 1988. – V. 127. – P. 279 – 308.

4. *Gorelov V.A.* On the algebraic independence of values of generalized hypergeometric functions // Mathematical Notes. – 2013. – V. 94, no. 1. – P. 82–95.

5. *Gorelov V.A.* On contiguity relations for generalized hypergeometric functions // Problemy Analiza – Issues of Analysis. – 2018, vol. 7(25), no. 2. – P. 39-46.

6. *Gorelov V.A.* On algebraic identities between solution matrices of Bessel's and Kummer's equations // Siberian Electronic Mathematical Reports. – 2019, vol. 16. – P. 258-262.

Kalman Győry. *Effective results for diophantine equations over finitely generated domains.*

In the first part of my talk, I give a brief survey of effective results and methods for Diophantine equations over finitely generated domains (FGD's) over Z which may contain transcendental elements, too. The first general results of this kind were obtained by Győry (1983, 1984) for decomposable form equations and discriminant equations. In 2013, Evertse and Györy combined Győry's method with an effective result of Aschenbrenner (2004) concerning ideal membership in polynomial rings over Z to establish effective finiteness therems for unit equations over FGD's over Z. Using their method, recently Bérczes, Evertse, Győry and Koymans proved such results for several important classes of polynomial and exponential Diohantine equations. In the second part of the talk I present some new effective results for equations of this type.

Lajos Hajdu. *Multiplicative decompositions of polynomial sequences.*

In the talk we present various new results concerning the asymptotic m-primitivity of sets of values of integer polynomials. Concerning arbitrary polynomials of degree $k \geq 3$, first we show that any infinite subset of the set of shifted k-th powers $\mathcal{M}'_k = \{1, 2, 2^k + 1, 3^k + 1, \ldots, x^k + 1, \ldots\}$ is totally m-primitive. Then we give a complete description of such polynomials whose value set in $\mathbb{N}$ is totally m-primitive. In the quadratic case we provide more precise statements. We also give a multiplicative analogue of a result of Sárközy and Szemerédi concerning changing elements of $\mathcal{M}'_k$ (related to a conjecture of Erdős), which is nearly sharp.

In our proofs we combine several tools from Diophantine number theory (including Pell equations, continued fractions, Baker's theory of Thue equations, the Bilu-Tichy method), a classical theorem of Wiegert on the number of divisors of positive integers and a theorem of Bollobás on the Zarankiewicz function related to an extremal problem for bipartite graphs. We conclude the talk with proposing some open problems.

The presented new results are joint with A. Sárközy.

Noriko Hirata-Kohno. *Linear forms in logarithms via Fel'dman's derivation method and the linear independence of polylogarithms.*

This is a joint work with Sinnou David and Makoto Kawashima.

Let K be an algebraic number field of finite degree over $\mathbb{Q}$. We first talk about Fel'dman's contribution in the theory of linear forms in logarithms by means of his derivation method. His method allows us to obtain a very sharp lower bound for linear forms in logarithms of non-zero algebraic numbers $\alpha_1, \cdots, \alpha_m$, with respect to the height of algebraic coefficients of the linear forms.

We present a refined lower bound via Padé approximation for the linear forms in logarithms, whenever $\alpha_1, \cdots, \alpha_m$ are sufficiently closed to 1.

Let $s \in \mathbb{Z}, s \geq 1$ and consider the s-th polylogarithm function $\mathrm{Li}_s(z)$ by

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

We discuss the linear independence of m polylogarithms, those are values of the polylogarithm function at $\alpha_1, \cdots, \alpha_m$, under several conditions including $\alpha_1, \cdots, \alpha_m$ are algebraic numbers which are very closed to 1.

Let us show a non-vanishing lemma of certain determinant, for proving the linear independence, of a matrix whose entries are related polynomials to polylogarithms. We explain how the determinant appears in the proof of the linear independence in Padé approximation.

Pavel Ivankov. *On Linear Independence of certain functions.*

Consider for $k=1,\dots,t,\ j=1,\dots,m+2$ hypergeometric functions

$$F_{kj}(z)=\sum_{\nu=0}^{\infty} z^{\nu}\chi_{kj}(\nu)\prod_{x=1}^{\nu}\frac{a(x)}{b(x)}\cdot\frac{1}{(x+\lambda_k)(x+A-\lambda_k)}. \tag{1}$$

In this equation $a(x)=(x+\alpha_1)\dots(x+\alpha_r),\ b(x)=(x+\beta_1)\dots(x+\beta_m),\ 0\leqslant r\leqslant m+1;\ \alpha_1,\dots,\alpha_r,$ $\beta_1,\dots,\beta_m,\ A,\lambda_1,\dots,\lambda_t$ — some complex numbers, and what's more

$$a(x)b(x)(x+\lambda_k)(x+A-\lambda_k)\neq 0 \tag{2}$$

for $x=1,2,\dots$;

$$\chi_{kj}(\nu)=\prod_{u=1}^{j-1}(\nu+\beta_u),\ k=1,\dots,t,\ j=1,\dots,m+1,\ \chi_{k,m+2}(\nu)=\chi_{k,m+1}(\nu)(\nu+\lambda_k).$$

Let $t,\ \tau_1,\dots,\tau_t$ be arbitrary natural numbers. Consider derivatives of the functions (1) with respect to parameter λ_k, i.e. the functions

$$F_{kl_kj}(z)=\sum_{\nu=0}^{\infty} z^{\nu}\chi_{kj}(\nu)\prod_{x=1}^{\nu}\frac{a(x)}{b(x)}\cdot\frac{\partial^{l_k}}{\partial\lambda_k^{l_k}}\prod_{x=1}^{\nu}\frac{1}{(x+\lambda_k)(x+A-\lambda_k)}, \tag{3}$$

where $k=1,\dots,t,\ l_k=0,1,\dots,\tau_k-1,\ j=1,\dots,m+2$.

Let condition (2) hold, and let

$$\alpha_i-\beta_j,\ \alpha_i-\lambda_k,\ \alpha_i+\lambda_k-A,\ i=1,\dots,r,\ j=1,\dots,m,\ k=1,\dots,t,$$

not be integers; let us suppose also that the numbers

$$\lambda_k-\lambda_{k'},\ \lambda_k+\lambda_{k'}-A,\ k\neq k',\ 2\lambda_k-A,\ k,k'=1,\dots,t.$$

are not integers. Then the functions (3) are linearly independent with the identically equal to unity function over the field $\mathbb{C}(z)$.

Vladimir Lysov. *On the irrationality and the irrationality exponent of the product of two logarithms*
.

We consider the Diophantine approximants for the product of the two logarithms $\gamma_m := \log\left(1+\frac{1}{m}\right)\log\left(1-\right.$ for an integer m. We prove that for all $m\geq 33$ the number γ_m is irrational. We also find new upper estimates of the irrationality exponent of γ_m, which tends to 4 as $m\to\infty$. This improves the previous results by M. Hata [1]. The irrationality of γ_m for large m was first proved by A. I. Galochkin [2].

Our approach is based on the Hermite–Padé approximants for the vector of functions (f_1,f_2,f_3), where

$$f_1(z):=\log\left(1+\frac{1}{z}\right),\quad f_2(z):=\log\left(1-\frac{1}{z}\right),\quad f_3:=f_1f_2.$$

This vector is an example of the Generalized Nikishin system of Markov functions on graphs [3, 4]. The common denominator of the approximants satisfies certain multiple orthogonality relations. The key ingredient of our proof is an explicit formula for the common denominator. By means of this formula we obtain the asymptotics of the sequence of the approximants and also some remarkable arithmetic properties of them.

References

[1] M. Hata. The irrationality of $\log(1+1/q)\log(1-1/q)$. *Trans. Amer. Math. Soc.* **350**:6 (1998), 2311–2327.

[2] A. I. Galochkin. Estimates from below of polynomials in the values of analytic functions of a certain class, *Mat. Sb.* **95(137)**:3(11) (1974), 396–417; *Math. USSR-Sb.* **24**:3 (1974), 385–407.

[3] V. N. Sorokin. Cyclic graphs and Apéry's theorem. *Russian Math. Surveys* **57**:3 (2002), 535–571.

[4] A. I. Aptekarev, V. G. Lysov. Systems of Markov functions generated by graphs and the asymptotics of their Hermite–Padé approximants. *Mat. Sb.* **201**:2 (2010), 183–234.

Raffaele Marcovecchio. *Hypergeometric rational approximations to $\zeta(4)$.*

This is a joint work with Wadim Zudilin. In 2003, Zudilin proposed a general construction of rational approximations to $\zeta(4)$, based on a certain very-well-poised integral, and showed that it leads to a conditional estimate for the irrationality exponent $\mu(\zeta(4))$ of the quantity $\zeta(4)$, subject to a suitable denominator conjecture. In the present work we circumvent that conjecture, by obtaining a new interpretation of those rational approximations. This leads to both unconditional and quantitatively better estimate for $\mu(\zeta(4))$.

Evgenii Matveev. *On numbers of bounded height from a fixed algebraic field.*

An upper estimate for a number of algebraic numbers of bounded height from a fixed algebraic fields is obtained together with an application to multiplicative dependence of algebraic numbers. . This result improves the analogous estimate of Loher-Masser of 2004.

Boris Moroz. *Diophantine equations (variations on the themes of Yu. V. Matiyasevich).*

I shall summarize my joint work with M.Carl on a Diophantine equation, encoding provability in formal mathematics and the recent work of A. Norkin on a Diophantine equation, whose insolubility is equivalent to the Riemann Hypothesis.

Arsenii Sagdeev. *On asymmetric Diophantine approximation.*

In 1891 Hurwitz proved his famous theorem about good rational approximations for a real number. This remarkable result can be formulated as follows.

Theorem. *Given an irrational number α, there are infinitely many rationals $\frac{p}{q}$ such that*

$$\frac{-1}{\sqrt{5}q^2} < \frac{p}{q} - \alpha < \frac{1}{\sqrt{5}q^2}.$$

One can define the set $\mathcal{H}$ of pairs of positive real numbers as follows.

$$\mathcal{H} = \left\{ (a,b) : \forall\, \alpha \in \mathbb{R}\backslash\mathbb{Q} \;\; \exists \text{ infinitely many } \;\; p,q \in \mathbb{Z} \text{ such that } \frac{-1}{aq^2} < \frac{p}{q} - \alpha < \frac{1}{bq^2} \right\}.$$

It is clear that each pair $(a, b) \in \mathcal{H}$ gives us an asymmetric analogue of Hurwitz's theorem. The problem of finding all the pairs $(a, b) \in \mathcal{H}$ was established by Segre in 1945 in different notation. Segre, Robinson, Tong and Alzer contributed to the solution but the problem is still unsolved.

Our new result generalizes all the previous and is close to the final solution. We described "almost all" boundary point of $\mathcal{H}$ and found out that the boundary is a fractal. We also make a conjecture that describes all the pairs $(a, b) \in \mathcal{H}$. We have not found the proof of this conjecture so far.

Vladislav Salikhov. *On estimates of simultaneous approximations of the rational numbers logarithms.*

In the lecture we consider three versions of integral constructions for investigation of diophantine approximations.

1. According the method from the article of K.Wu, 2002, were obtained new lower boundary estimates for linear forms in $1, \log 2, \log 3, \log 5$, and also in $1, \log 2, \log 3, \log 5, \log 7$.

2. By applying a symmetrized version of R.Marcoveccio integral we get a new estimate of approximation π by numbers from the field $\mathbb{Q}(\sqrt{3})$.

3. Using symmetrized integrals, we get estimate for linear form in $1, \log 2, \log 3, \log 7$, which gives a new result for irrationality measure of the value of $\log 7$. This method will be considered in more details in the lecture by Bashmakova.

Ilya Shkredov. *Some progress in the problem of Zaremba.*

This is joint work with N.G. Moshchevitin and B. Murphy.

Zaremba's famous conjecture posits that there is an absolute constant M with the following property: for any positive integer q there exists a number a coprime to q such that in the continued fraction expansion of a/q all partial quotients are bounded by M. Although this conjecture is widely open, a series of different results in the direction were obtained by Korobov, Niedereiter, Bourgain, Kontorovich, Frolenkov, Kan and others Using methods of Additive Combinatorics (we apply growth results in $SL_2(\mathbf{F}_p)$), we obtain a sharp upper bound for cardinality of Zaremba's numbers a, i.e. the set of a such that Zaremba's hypothesis holds.

Nikita Shulga. *Diophantine properties of fixed points of Minkowski question mark function.*

The Minkowski question-mark function ?(x) is a continuous and strictly increasing function, which maps the $[0, 1]$ interval onto itself. If $[0; a_1, a_2, \dots, a_n, \dots]$ is the continued fraction representation of an irrational number x, then

$$?(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+\dots+a_k-1}}$$

If x is a rational number, then the sum in the formula behind is replaced by a finite sum. One can easily see that the equation $?(x) = x$ has at least 5 solutions, a folklore conjecture states that there are exactly 5 fixed points of $?(x)$. We will tell about some new results on Diophantine properties of fixed points of $?(x)$. We also give a condition from which the conjecture about the fixed points would follow.

The talk is based on a joint work with Dmitry Gayfulin.

Carlo Viola. *The permutation group method: results and problems.*

The permutation group method in Diophantine approximation was introduced by G. Rhin and myself in 1996 and was applied, since then, to obtain several qualitative and quantitative results of irrationality and of linear independence over the rationals. I will present a short survey of the method, and sketch some forthcoming new applications.

Paul Voutier. *Sharp bounds on the number of solutions of* $X^2 - (a^2 + b)\, Y^4 = -b$.

In [SWY], Stoll, Walsh and Yuan showed that the diophantine equation

$$X^2 - \left(1 + 2^{2m}\right) Y^4 = -2^{2m}$$

has at most three solutions in positive integers (X, Y).

In this talk, we present new results that are an improvement and substantial generalization of their result.

Let a, m and p be non-negative integers with $a \geq 1$ and p a prime. For $b = p^m$, $2p^m$ or $4p^m$, suppose that $a^2 + b$ is not a square and that $x^2 - (a^2 + b)\, y^2 = -4$ has an integer solution.

If b is a square, then there are at most two coprime positive integer solutions of

$$X^2 - \left(a^2 + b\right) Y^4 = -b.$$

If b is not a square, then there are at most three coprime positive integer solutions. Moreover, these results are best possible.

In the course of the proof, we acturally prove results for arbitrary values of b under suitable conditions for solutions of $x^2 - (a^2 + b)\, y^2 = -b$.

Our proof is based on a novel use of the hypergeometric method that may also be useful for other problems.

References.

[SWY] M. Stoll, P. G. Walsh and P. Yuan, The Diophantine equation $X^2 - (2^{2m} + 1)\, Y^4 = -2^{2m}$ II, *Acta Arith.* **139** (2009), 57–63.

Michel Waldschmidt. *Integer valued entire functions and Lidstone series.*

In this talk, we connect two different topics. One is the theory of integer valued entire functions, a subject of mathematics where A.O. Gel'fond had important contributions. The other is the theory of Lidstone series, which is a variant of Taylor series for the approximation of an analytic function by a series involving the derivatives of even order of the function at two points. Here is an example of our results. If the derivatives of even order at 0 and 1 of an entire function are integers and if the function has exponential type < 1, then it is a polynomial. We prove that our estimate is best possible by producing an uncountable set of examples.

Daria Zabrodina. *On an identity for hypergeometric type integrals.*

Introduction. Let X be a linear space over $\mathbb{Q}$, generated by an empty word and all words, composed from the letters x_0, x_1. For a word $w \in X$ we determine its weight $|w|$ - a number of all letters in this word.

Denote

$$w_0(z) = \frac{1}{z},\ w_1(z) = \frac{1}{1-z}.$$

The *generalized polylogarithm*, corresponding to a word $w = x_{\varepsilon_1} \dots x_{\varepsilon_{k-1}} x_1 x_0^m$, where m is a non-negative integer and ε_i is equal 0 or 1, $i = 1, \dots, k-1$, is defined as follows:

$$Li_w(z) = \int\limits_0^z w_{\varepsilon_1}(z_1)dz_1 \int\limits_0^{z_1} w_{\varepsilon_2}(z_2)dz_2 \dots \int\limits_0^{z_{k-1}} \frac{1}{1-z_k}\frac{\ln^m(z_k)}{m!}dz_k.$$

We extend the definition of polylogarithms to space X by linearity.

Polylogarithms at $\frac{-z}{1-z}$. It was shown that the value of polylogarithms at point $\frac{-z}{1-z}$ can be expressed as a linear combination of polylogarithms at point z [1]:

Theorem 1. *Let $a_1, \dots, a_k$ be a set of non-negative integers, $z \in \mathbb{C} : |z| < 1$, $|\frac{-z}{1-z}| < 1$. Then*

$$Li_{x_0^{a_1}x_1 \dots x_0^{a_k}x_1}\Big(\frac{-z}{1-z}\Big) = (-1)^k \sum_{\substack{|u_1|=a_1 \\ \dots \\ |u_k|=a_k}} Li_{u_1x_1\dots u_kx_1}(z).$$

We give another proof of this fact based on a change of variables in the integral.

Let $w = x_{\varepsilon_1}x_{\varepsilon_2}\dots x_{\varepsilon_{n-1}}x_1$. Then

$$Li_w\Big(\frac{-z}{1-z}\Big) = \int\limits_0^{\frac{-z}{1-z}} w_{\varepsilon_1}(z_1)dz_1 \int\limits_0^{z_1} w_{\varepsilon_2}(z_2)dz_2 \dots \int\limits_0^{z_{n-1}} w_1(z_n)dz_n.$$

We make the substitution $z_1 = \frac{-y_1}{1-y_1}, \dots, z_n = \frac{-y_n}{1-y_n}$:

$$\text{then } w_0\Big(\frac{-y_i}{1-y_i}\Big)dz_i = (w_0(y_i) + w_1(y_i))dy_i$$

$$\text{and } w_1\Big(\frac{-y_i}{1-y_i}\Big)dz_i = -w_1(y_i)dy_i.$$

Denoting $v_0(z) = w_0(z) + w_1(z), v_1(z) = -w_1(z)$, we have

$$Li_w\Big(\frac{-z}{1-z}\Big) = \int\limits_0^{z} v_{\varepsilon_1}(y_1)dy_1 \int\limits_0^{y_1} v_{\varepsilon_2}(y_2)dy_2 \dots \int\limits_0^{y_{n-1}} v_1(y_n)dy_n.$$

To prove the theorem, we note that after integrating functions v_1, which correspond to x_1 in w, we obtain factor -1:

$$\int\limits_0^{y_{i-1}} v_1(y_i)Li_u(y_i)dy_i = -Li_{x_1u}(y_{i-1}).$$

And after integrating functions v_0, which correspond to x_0 in w, we obtain the sum of polylogarithms:

$$\int\limits_0^{y_{i-1}} v_0(y_i)Li_u(y_i)dy_i = Li_{x_0u}(y_{i-1}) + Li_{x_1u}(y_{i-1}).$$

Hypergeometric type integrals. Denote

$$\overline{a} = (a_1, \dots, a_m), \overline{b} = (b_1, \dots, b_m), \overline{c} = (c_1, \dots, c_m).$$

The symbol $\{a\}_k$ stands for a sequence that consists of number a taken k times.
Let $0 = r_0 < r_1 < \ldots < r_{l-1} < r_l = m$ and $c_i = 0$, if and only if $i \notin \{r_1, \ldots, r_l\}$, $i = 1 \ldots m$, so that

$$\overline{c} = \left(\{0\}_{r_1-r_0-1}, c_{r_1}, \ldots, \{0\}_{r_l-r_{l-1}-1}, c_{r_l}\right).$$

We define

$$I_m(\overline{a};\overline{b};\overline{c}|z) = z^{a_1}\prod_{i=1}^{m}\frac{1}{\Gamma(b_i-a_i)}\int\limits_{[0,1]^m}\cdots\int\frac{\prod_{i=1}^{m}x_i^{a_i-1}(1-x_i)^{b_i-a_i-1}}{\prod_{j=1}^{l}\left(1-zx_1\cdots x_{r_j}\right)^{c_{r_j}}}dx_1\cdots dx_m,$$

where it is supposed that $z^{a_1} = e^{a_1(\ln|z|+i\cdot\arg(z))}$.

E. A. Ulanskii proved the following theorem [2]:

Theorem 2. *Let $z, a_i, b_i, c_i \in \mathbb{C}$, $|arg(1-z)| < \pi$ and $Re(a_i) > 0$, $b_i - a_i \in \mathbb{N}, i = 1, \ldots, m$. Then*

$$I_m\Big(\overline{a};\overline{b};\overline{c}|\frac{-z}{1-z}\Big) = (-1)^{a_1}I_m(\overline{a};\overline{b};\overline{b}-\overline{a}'-\overline{c}|z),$$

where $\overline{a}' = (a_2, \ldots, a_m, 0)$.

We will generalize the previous technique and prove this statement by making several changes of variables in the integrals. Denote $D = \prod_{i=1}^{m}\frac{1}{\Gamma(b_i-a_i)}$.

Variable change 1:

Let $y_1 = zx_1, y_2 = zx_1x_2, \ldots, y_m = zx_1\ldots x_m$. Hence, we obtain $I_m(\overline{a};\overline{b};\overline{c}|z) =$

$$= D\int_0^z\int_0^{y_1}\cdots\int_0^{y_{m-1}}\frac{\prod_{i=1}^{m-1}y_i{}^{a_i-a_{i+1}-1}y_m{}^{a_m-1}\left(1-\frac{y_1}{z}\right)^{b_1-a_1-1}\prod_{i=2}^{m}\left(1-\frac{y_i}{y_{i-1}}\right)^{b_i-a_i-1}}{\prod_{j=1}^{l}\left(1-y_{r_j}\right)^{c_{r_j}}}dy_1\cdots dy_m.$$

Variable change 2:

We make the substitution $y_i = \frac{-t_i}{1-t_i}, i = 1, \ldots, m$, then $dy_i = \frac{-1}{(1-t_i)^2}dt_i$. It implies that

$$I_m(\overline{a};\overline{b};\overline{c}|z) = (-1)^{a_1}D\int_0^{\frac{-z}{1-z}}\int_0^{t_1}\cdots\int_0^{t_{m-1}}\frac{\prod_{i=1}^{m-1}t_i{}^{a_i-a_{i+1}-1}\,t_m{}^{a_m-1}\prod_{j=1}^{l}(1-t_{r_j})^{c_{r_j}}}{\prod_{i=1}^{m-1}(1-t_i)^{a_i-a_{i+1}+1}\,(1-t_m)^{a_m+1}}\times$$

$$\times\left(1+\frac{t_1}{z(1-t_1)}\right)^{b_1-a_1-1}\prod_{i=2}^{m}\left(1-\frac{t_i(1-t_{i-1})}{(1-t_i)t_{i-1}}\right)^{b_i-a_i-1}dt_1\cdots dt_m.$$

Variable change 3: Let $s = \frac{-z}{1-z}$ and $t_1 = sx_1, t_2 = sx_1x_2, \ldots, t_m = sx_1\ldots x_m$. Therefore, $I_m(\overline{a};\overline{b};\overline{c}|z) =$

$$= (-1)^{a_1}s^{a_1}D\int\limits_{[0,1]^m}\cdots\int\frac{\prod_{i=1}^{m}x_i^{a_i-1}(1-x_i)^{b_i-a_i-1}\prod_{j=1}^{l}(1-sx_1\cdots x_{r_j})^{c_{r_j}}}{\prod_{i=1}^{m-1}(1-sx_1\cdots x_i)^{b_i-a_{i+1}}\,(1-sx_1\cdots x_m)^{b_m}}dx_1\cdots dx_m =$$

$$= (-1)^{a_1}I_m(\overline{a};\overline{b};\overline{b}-\overline{a}'-\overline{c}|\frac{-z}{1-z}).$$

This concludes the proof of Theorem 2.

References.

[1] Ulanskii E. A. *Identities for Generalized Polylogarithms.*

Mathematical Notes, vol. 73, no. 4, 2003, pp. 571-581.
[2] Ulanskii E. A. *An identity for generalizing the hypergeometric integral.* Mathematical Notes, vol. 79, no. 5, 2006, pp. 741-744.

Wadim Zudilin. *Many odd zeta values are irrational.*

It is known that all even zeta values $\zeta(2k)$, where $k = 1, 2, \dots$, are irrational and, thanks to Apéry (1978), that $\zeta(3)$ is irrational. A theorem of Rivoal and Ball (2000) demonstrates that infinitely many of odd zeta values $\zeta(2k+1)$, where $k = 1, 2, \dots$, happen to be irrational; more precisely, the theorem tells that of the numbers $\zeta(5), \zeta(7), \dots, \zeta(2k+1)$ at least $c \log k$ are irrational, for some absolute constant $c > 0$. In our recent work with Fischler and Sprang we produce a new construction of simultaneous $\mathbb{Q}$-linear forms in odd zeta values and replace the Ball–Rivoal bound $c \log k$ with $\exp\{(c \log k)/(\log\log k)\}$, something "much more like a power of k than a power of $\log k$" according to Hardy and Wright. I shall highlight principal ingredients of this construction, mainly explaining how to prove that one of $\zeta(5), \zeta(7), \dots, \zeta(2k+1)$ for some k is irrational by elementary means.