Possible profiles of Sperner families

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Let $[n] = \{1, 2, \dots, n\}$ be our underlying set. We will consider families $\mathcal{F} \subset 2^{[n]}$ of subsets of $[n]$. Sperner's theorem claims that the maximum of $|\mathcal{F}|$ is $\binom{n}{\lfloor n/2 \rfloor}$ under the condition that $\mathcal{F}$ contains no pair of distinct members F, G such that $F \subset G$. (Such families will be called *Sperner families* in what follows.) The so-called YBLM-inequality

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$$

also holds for Sperner families. This is an obvious strengthening of Sperner's theorem. Let $\mathcal{F}_i$ be the subfamily of $\mathcal{F}$ consisting of its i-element members. Introducing the notation $f_i = f_i(\mathcal{F}) = |\mathcal{F}_i|$ the YBLM-inequality can be rewritten in the following form.

$$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \leq 1.$$

Call the vector $(f_0(\mathcal{F}), f_1(\mathcal{F}), \dots, f_n(\mathcal{F}))$ the *profile vector* of $\mathcal{F}$. The inequality expresses that the profile vectors of Sperner families are below the corresponding hyperplane. Bey gave a strengthening: a stronger quadratic inequality having the same property. Griggs and the author improved Bey's inequality by a system of pieces of hyperplanes.