Some research problems for students

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1 Hyperbolic Reflection Groups: Introduction

Let $\mathbb{X}^n$ be one of the three spaces of constant curvature, that is, either the Euclidean space $\mathbb{E}^n$, or the n-dimensional sphere $\mathbb{S}^n$, or the n-dimensional (hyperbolic) Lobachevsky space $\mathbb{H}^n$.

Consider a convex polytope P in the space $\mathbb{X}^n$. If we act on P by the group Γ generated by reflections in the hyperplanes of its faces it can occur that the images of this polyhedron corresponding to different elements of Γ will cover the entire space $\mathbb{X}^n$ and will not overlap with each other. In this case we say that Γ is a *discrete reflection group*, and the polytope P is *the fundamental polyhedron* for Γ. If the polytope P is *bounded* (or, equivalently, *compact*), then the group Γ is called *a cocompact reflection group*, and if the polytope P has a *finite volume*, then the group Γ is called *cofinite* or a discrete group of *finite covolume*.

Which properties characterize such polyhedra P? For example, any two hyperplanes H_i and H_j bounding P either do not intersect or form a dihedral angle equal to π/n_{ij}, where $n_{ij} \in \mathbb{Z}$, $n_{ij} \geq 2$.

Such polyhedra are called *Coxeter polyhedra*, since the discrete reflection groups of finite covolume (hence their finite volume fundamental polyhedra) for $\mathbb{X}^n = \mathbb{E}^n, \mathbb{S}^n$ were determined and found by H.S.M. Coxter in 1933.

In 1967, E. B. Vinberg developed his theory of discrete groups generated by reflections in the Lobachevsky spaces. He proposed new methods for studying hyperbolic reflection groups, in particular, a description of such groups in the form of the so-called Coxeter–Vinberg diagrams. He formulated and proved the arithmeticity criterion for hyperbolic reflection groups and constructed a number of various examples.

2 Reflective Hyperbolic Lattices: Preliminaries

Suppose $\mathbb{F}$ is a totally real number field with the ring of integers $A = \mathbb{O}_{\mathbb{F}}$. Suppose $f(x)$ is a quadratic form of signature $(n, 1)$ defined over $\mathbb{F}$ such that for every non-identity embedding $\sigma\colon \mathbb{F} \to \mathbb{R}$ the form f^σ is positive definite. Such forms $f(x)$ are said to be admissible.

A free finitely generated A-module L with an inner product of signature $(n, 1)$ is said to be *a hyperbolic lattice* if for each non-identity embedding $\sigma\colon \mathbb{F} \to \mathbb{R}$ the quadratic space $L \otimes_{\sigma(A)} \mathbb{R}$ is positive definite. (An inner product in L is associated with some admissible quadratic form.)

Suppose $O'(L)$ is the group of integral automorphisms of a lattice L preserving the n-dimensional Lobachevsky (hyperbolic) space $\mathbb{H}^n$.

A primitive vector $e \in L$ is called a *root* or, more precisely, a *k-root* where $k = (e, e) > 0$ if $2(e, x) \in kA$ for all $x \in L$. For some numbers k the latter condition holds automatically. We will call such numbers *stable*. For example, if $A = \mathbb{Z}$ then stable numbers are $k = 1$ and $k = 2$. For $A = \mathbb{Z}[\sqrt{2}]$ stable numbers are $k = 1, 2, 2 + \sqrt{2}$.

Any root e defines an orthogonal reflection $R_e\colon x \mapsto x - \dfrac{2(e,x)}{(e,e)} e$, which preserves the lattice

L. This reflection is called the *k-reflection* if $(e, e) = k$. In the hyperbolic case, R_e defines a reflection of $\mathbb{H}^n$ with respect to the invariant hyperplane $H_e = \{x \in \mathbb{H}^n : (x, e) = 0\}$, which is called a mirror of reflection R_e. For the stable number k the corresponding k-reflection is called a *stable reflection*.

If the subgroup $O_r(L)$ of $O'(L)$ generated by all reflections is of finite index, then the lattice L is said to be *reflective*. It is equivalent to the fact that the fundamental (Coxeter) polyhedron of the group $O_r(L)$ has a finite volume in the space $\mathbb{H}^n$. We can define in the same way the subgroup $S(L)$ generated by all *stable reflections*. If $S(L)$ is a finite index subgroup of $O'(L)$ then the lattice L is called *stably reflective*.

All the groups of type $O_r(L)$ (or $S(L)$) as well as all the reflection groups commensurable with them are called *arithmetic hyperbolic reflection groups* with the *field of definitions* (or the *ground field*) $\mathbb{F}$.

3 Open problems

3.1 Big open problems

Our discussion above leads us to the following fundamental open problems connected with the theory of discrete reflection groups and Coxeter polytopes in the Lobachevsky spaces $\mathbb{H}^n$.

Problem 1. *Constructing of arithmetic and non-arithmetic hyperbolic reflection groups.*

Problem 2. *Which is the maximal dimension of the Lobachevsky space in which there exist compact Coxeter polytopes? A similar question is open for Coxeter polytopes of finite volume.*

Problem 3. *Classification of hyperbolic reflection groups of finite covolume, classification of reflective hyperbolic lattices, and classification of maximal arithmetic hyperbolic reflection groups.*

Remark 1. *The problem of classification of reflective hyperbolic lattices was actually posed in cited work of Vinberg in 1967. Further results obtained in the 1970-80s (and also some recent results) definitely confirm that there is a hope to solve these problems.*

A very efficient tool for solving problems 1 and is Vinberg's Algorithm (1972) of constructing the fundamental polyhedron for a hyperbolic reflection group. Practically it is efficient for arithmetic reflection groups. It enables one given a lattice to determine if this lattice is reflective.

3.2 Partial problems

Problem 4. *Efficient software implementation of Vinberg's algorithm for hyperbolic lattices over fields $\mathbb{Q}$ and at least $\mathbb{Q}[\sqrt{d}]$ for some small numbers d.*

Problem 5. *Classification of stably reflective hyperbolic $\mathbb{Z}$-lattices of rank > 3.*

Problem 6. *Classification of stably reflective hyperbolic $\mathbb{Z}[\sqrt{2}]$-lattices of rank > 3.*

Problem 7. *Classification of reflective hyperbolic $\mathbb{Z}$-lattices of rank > 3 with orthogonal basis.*

Problem 8. *Classification of arithmetic and non-arithmetic Coxeter polytopes with some simple combinatorial structure (prisms, pyramids, etc).*

4 Known results

4.1 Problems 1 and 2

The record example of a compact Coxeter polyhedron was found by V.O. Bugaenko for $n = 8$, although the maximal possible dimension is bounded by the inequality $n < 30$.

A record example of a Coxeter polyhedron of finite volume belongs to R. Borcherds in the dimension $n = 21$. It is known that the Coxeter polytopes of finite volume can exist only for $n < 996$ (see papers of M. Prokhorov and A. Khovanskii, 1986).

Both examples came from arithmetic reflection groups. Bugaenko's example is the fundamental polyhedron for some arithmetic reflection group over the field $\mathbb{Q}[\sqrt{2}]$ in the space $\mathbb{H}^8$, the Borcherds example is the fundamental polyhedron for some arithmetic group of reflections over a field $\mathbb{Q}$ in the space $\mathbb{H}^{21}$.

Moreover, D. Allcoc, using an elegant and a simple doubling trick, has constructed infinite series of finite volume Coxeter polytopes in Lobachevsky spaces through dimension 19, and also of compact Coxeter polytopes through dimension 6. We also note that in dimensions 7 and 8 they can be taken to be either arithmetic or nonarithmetic.

4.2 Problem 3

As for the third problem, it is also far from being completely solved. An effective description of all discrete reflection groups in the spaces $\mathbb{H}^n$ is obtained only for $n = 2$ (H. Poincaré, 1882) and for $n = 3$ (the famous theorems of E. M. Andreev, 1970).

In the classification of arithmetic hyperbolic reflection groups a more significant success has been achieved. Over the definition field $\mathbb{Q}$, the reflective hyperbolic lattices are classified for $n = 2$ (V.V. Nikulin, 2000 and D. Allcock, 2011), $n = 4$ (R. Sharlau and C. Walhorn, 1989–1993), $n = 5$ (I. Turkalj, 2017) and in the noncompact (isotropic) case for $n = 3$ (R. Sharlau and C. Walhorn, 1989–1993).

A classification of reflective hyperbolic lattices of signature $(2, 1)$ with the definition field $\mathbb{Q}[\sqrt{2}]$ was obtained by A. Mark in 2015.

In all other cases, Problem 3 remains open.

4.3 Problem 4

A. Perepechko and I made a software implementation of Vinberg's Algorithm for Integral Hyperbolic Lattices (see [3] and [4]). Our computer program supports lattices with non-diagonal basis. It works efficiently enough in dimensions $2 \leq n \leq 6$. Also we hope to made a general implementation of Vinberg's Algorithm for lattices over quadratic fields of type $\mathbb{Q}[\sqrt{d}]$.

Recently I made some computer programs for some lattices over $\mathbb{Z}[\sqrt{2}]$.

4.4 Problem 5–8

There are known only classifications of stably reflective hyperbolic lattices of rank 4 over $\mathbb{Z}$ and $\mathbb{Z}[\sqrt{2}]$, see my papers [1, 2, 5, 6] and my PhD-thesis [7].

5 Conclusion

Problems 4–8 could be very good problems as research problems under my supervision. They could be very good problems for Bachelor's or Master's thesis. There are known many ideas and methods for trying to solve it.

References

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