Research problems for students

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Notations: $[n] := \{1, \dots, n\}$.

Problem 1. Let $v_1, \dots, v_n$ be unit distinct vectors in $\mathbb{R}^d$ such that among any three of them, there are two orthogonal and $\langle v_i, v_j \rangle \geq 0$ for $i, j \in [n]$. Prove that $n \leq \lfloor 3d/2 \rfloor$.

See [Ros91, Pol17].

Problem 2. Let $v_1, \dots, v_n$ be unit distinct vectors in $\mathbb{R}^d$ such that there are $a, b \in [-1, 1)$ such that for distinct $i, j \in [n]$, we have $\langle v_i, v_j \rangle = a$ or $\langle v_i, v_j \rangle = b$. Also, assume that among any three vectors, there are two with the scalar product equal to a. Prove that $n = O(d)$.

Some known results can be found in [Pol17].

Problem 3. Let S_1, S_2, and S_3 be three basis of $\mathbb{R}^d$. Prove that there are three pairwise non-orthogonal vectors $u_1 \in S_1, u_2 \in S_2$, and $u_3 \in S_3$.

Problem 4. Goodman-Goodman Spherical Problem. Assume that $s_1, \dots, s_n$ are caps on a unit sphere and $\alpha_1, \dots, \alpha_n$ are their spherical radii respectively. Let $\alpha_1 + \dots + \alpha_n < \pi/2$. Prove that either there is a cap of radius $\alpha_1 + \dots + \alpha_n$ covering $s_1, \dots, s_n$ or there is a plane passing through the center of the unit sphere such that it does not intersect $s_1, \dots s_n$ and separates them into two non-empty parts.

Note that the analogous fact was proved in the plane [GG45]. The case $\alpha_1 + \dots + \alpha_n = \pi/2$ is solved in [JP17].

References

[GG45] A. Goodman and R. Goodman, *A circle covering theorem*, The American Mathematical Monthly **52** (1945), no. 9, 494–498.

[JP17] Z. Jiang and A. Polyanskii, *Proof of László Fejes Tóth's zone conjecture*, Geometric and Functional Analysis **27** (2017), no. 6, 1367–1377. Available at https://arxiv.org/abs/1703.10550.

[Pol17] A. Polyanskii, *On almost-equidistant sets-II* (2017). Available at https://arxiv.org/abs/1708.02039.

[Ros91] M. Rosenfeld, *Almost orthogonal lines in E^d*, Applied geometry and discrete mathematics, DIMACS: Series in Discrete Mathematics and Theoretical Computer Science, vol. 4, American Mathematical Society, Providence, RI, 1991, pp. 489–492.

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