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Logical limit laws and logical complexity

Studying logical laws of random graphs requires an amount of logical prerequisites. We review some of the basics here, and refer the reader to [1, 2, 3]. Sentences in the first order language of graphs (FO sentences) are constructed using relational symbols $\sim$ (interpreted as adjacency) and $=$, logical connectives $\neg, \rightarrow, \leftrightarrow, \vee, \wedge$, variables $x, y, x_1, \ldots$ that express vertices of a graph, quantifiers $\forall, \exists$ and parentheses (,). Monadic second order, or MSO, sentences are built of the above symbols of the first order language, as well as the variables $X, Y, X_1, \ldots$ that are interpreted as unary predicates, i.e. *subsets* of the vertex set. In an MSO sentence, variables $x, y, x_1, \ldots$ (that express vertices) are called *FO variables*, and variables $X, Y, X_1, \ldots$ (that express sets) are called *MSO variables*. If, in an MSO sentence φ, all the MSO variables are existential and in the beginning (that is $\varphi = \exists X_1 \ldots \exists X_m\ \phi(X_1, \ldots, X_m)$ where $\phi(X_1, \ldots, X_m)$ is a first order sentence with unary predicates $X_1, \ldots, X_m$), then the sentence is called *existential* monadic second order (EMSO). Sentences must have finite length.

We call the number of nested quantifiers in a longest sequence of nested quantifiers of a sentence φ *the quantifier depth* $\mathrm{q}(\varphi)$. For example, the MSO sentence

$$\forall X \quad \Big([\exists x_1 \exists x_2\ X(x_1) \wedge \neg X(x_2)] \rightarrow [\exists y \exists z\ X(y) \wedge \neg X(z) \wedge y \sim z] \Big)$$

has quantifier depth 3 and expresses the property of being connected (and its first order part has quantifier depth 2). It is known that the property of being connected *cannot be expressed* by a FO sentence. This fact (and many other facts about an expressibility) may be easily proved using Ehrenfeucht–Fraïssé games (see e.g., [1, 3]).

The quantifier depth of a sentence has the following clear algorithmic interpretation: an FO sentence of quantifier depth k on an n-vertex graph can be verified in $O(n^k)$ time. It is very well known (see, e.g., [1], Proposition 6.6) that the same is true for the number of variables: an FO sentence with k variables on an n-vertex graph can be verified in $O(n^k)$ time. The later statement is stronger because, clearly, every FO sentence of quantifier depth k may be rewritten using at most k variables.

In what follows, for a sentence φ, we use the usual notation from model theory $G \models \varphi$ if φ is true for G.

Let F be an arbitrary graph, and $D(F)$ (resp. $W(F)$) be the minimum quantifier depth (resp. the minimum number of variables) of a FO sentence expressing the property of containing a subgraph isomorphic to F. It is straightforward that $D(F) = W(F) = v(F)$, where $v(F)$ denotes the number of vertices of F. Unfortunately, in algorithmic sense, it does not give anything non-trivial. However, the *induced* case differs a lot. Let $D[F]$ (resp. $W[F]$) be the minimum quantifier depth (resp. the minimum number of variables) of a FO sentence expressing the property of containing an **induced** subgraph isomorphic to F. It is known that there exists a graph F on 4 vertices such that $D[F] = W[F] = 3$. Moreover, there are infinitely many graphs with $W[F] < v(F)$.

Question 1. For an arbitrary graph F, find $D[F]$ and $W[F]$.

In 1959, P. Erdős and A. Rényi, and independently E. Gilbert, introduced two closely related models for generating random graphs. A seminal paper of Erdős and Rényi, that appeared one year later, brought a lot of attention to the subject, giving birth to Erdős-Rényi random graphs. In spite of the name, the more popular model $G(n,p)$ is the one proposed by Gilbert. In this model, we have $G(n,p) = (V_n, E)$, where $V_n = \{1,\ldots,n\}$, and each pair of vertices is connected by an edge with probability p and independently of other pairs. For more information, we refer readers to the books [4, 5, 6]. Y. Glebskii, D. Kogan, M. Liogon'kii and V. Talanov in 1969, and independently R. Fagin in 1976, proved that any FO sentence is either true with asymptotical probability 1 (asymptotically almost surely or a.a.s.) or a.a.s. false for $G(n, 1/2)$, as $n \to \infty$. In such a situation we say that $G(n,p)$ *obeys the FO zero-one law*. More generally, consider a logic $\mathcal{L}$. We say that $G(n,p)$ *obeys the* $\mathcal{L}$ *zero-one law* if, for every sentence $\varphi \in \mathcal{L}$, $\lim_{n\to\infty} \mathsf{P}(G(n,p) \models \varphi) \in \{0,1\}$. A weaker version of this law is called the convergence law: $G(n,p)$ *obeys the* $\mathcal{L}$ *convergence law* if, for every sentence $\varphi \in \mathcal{L}$, the limit $\lim_{n\to\infty} \mathsf{P}(G(n,p) \models \varphi)$ exists (but not necessarily equals 0 or 1).

Remark. J. Spencer was the first who noticed that Ehrenfeucht–Fraïssé games may be applied for proofs of zero-one laws: if, for any number of rounds, a.a.s. there exists a winning strategy (in the game w.r.t. the logic $\mathcal{L}$) of Duplicator on two independent random graphs $G(n,p)$ and $G(m,p)$ (as $n, m \to \infty$), then the zero-one law for sentences from $\mathcal{L}$ holds.

However, $G(n, 1/2)$ does not obey even the convergence law for EMSO. The respective construction φ was obtained by J.-M. Le Bars in 2001 [7]. The first MSO sentence with one binary relation that has no asymptotic probability was constructed by M. Kaufmann and S. Shelah in 1985. In 1987, Kaufmann proved that there exists an EMSO sentence with 4 binary relations that has no asymptotic probability. Note that this construction contains 4 monadic variables and 9 first order variables, and the sentence φ proposed by Le Bars has even more variables (of both types). In the above mentioned paper, Le Bars conjectured that, for EMSO sentences with 2 first order variables, $G(n, 1/2)$ obeys the zero-one law.

Recently, S. Popova and M. Zhukovskii disproved the conjecture for $G(n, 1/2)$ (the link: https://arxiv.org/pdf/1807.01794.pdf). Moreover, first, they proved that the minimum FO quantifier depth of a EMSO sentence without convergence equals 2, and the same is true for the number of FO variables. Second, there exists an EMSO sentence with only one monadic variable and without convergence. Surprisingly, 2 FO variables are enough in this case as well! However, the minimum FO quantifier depth of an EMSO sentence with 1 monadic variable and without convergence equals 3. Finally, in this paper it is proven that there exists a dense subset $\mathcal{P} \in (0, 1)$ such that, for every $p \in \mathcal{P}$, the Le Bars conjecture is false as well.

Question 2. Is there any $p \in (0, 1)$ such that $G(n, p)$ obeys 0-1 law for EMSO sentences with 2 FO variables?

In 1988, S. Shelah and J. Spencer proved that $G(n, n^{-\alpha})$ obeys the FO 0-1 law if and only if α is either irrational or bigger than 1 and does not equal to any number $1 + 1/m$, $m \in \mathbb{N}$. For a FO sentence ϕ, let $S(\phi)$ be the set of all $\alpha > 0$ such that $\mathsf{P}(G(n, n^{-\alpha}) \models \phi)$ does not converge neither to 0, nor to 1. Clearly $S(\phi) \subset \left(\mathbb{Q} \cap (0, 1]\right) \cup \{1 + 1/m,\ m \in \mathbb{N}\}$. In 1990, J. Spencer proved that there exists a FO sentence such that $S(\phi)$ is infinite. I have proved that the minimum quantifier depth of a FO sentence ϕ with an infinite $S(\phi)$ is either 4 or 5.

Question 3. What is the minimum number of variables of a FO sentence ϕ with an infinite $S(\phi)$?

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