On the list chromatic number of the complete multi-partite hypergraphs and multiple coverings by independent sets

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Abstract

The paper deals with list colorings of hypergraphs. Consider the hypergraph $H(m, r, k)$, the complete r-partite k-uniform hypergraph with parts of equal size m, in which any edge takes exactly one vertex from some $k < r$ parts. Using the results concerning multiple coverings by independent sets we prove that for fixed k and r, the list chromatic number of $H(m, r, k)$ is equal to $(1 + o(1)) \log_{\frac{r}{r-k+1}}(m)$ as $m \to \infty$.

Keywords: list colorings, colorings of hypergraphs, independent sets.

1 Introduction and the background of the problem

The work is devoted to the list colorings of uniform hypergraphs. Let us recall some definitions.

Let $H = (V, E)$ be a hypergraph. A *vertex coloring* f is a mapping from the vertex set V to some set of colors C. A coloring is called *proper* for H if there is no monochromatic edges in E under it, i.e. formally, for every edge $A \in E$,

$$|\{f(v) : v \in A\}| > 1.$$

The chromatic number of H, $\chi(H)$, is the minimum s such that there is proper coloring for H with s colors.

The extension of the chromatic number is the notion of the list chromatic number. A hypergraph $H = (V, E)$ is said to be *s-choosable* if for every family of sets $L = \{L(v) : v \in V\}$ (L is called *the list assignment*), such that $|L(v)| = s$ for all $v \in V$ (*s-uniform* list assignment), there is a proper coloring from the lists, i.e. for every $v \in V$, we should use a color from

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$L(v)$. The *list chromatic number* of H, denoted by $\chi_l(H)$, is the minimum s such that H is s-choosable.

The study of list colorings of graphs and hypergraphs was initiated by Vizing [1] and by Erdős, Rubin and Taylor [2]. One of the first interesting results concerning the list chromatic number is that it can be much larger than the usual chromatic number. In particular, Erdős, Rubin and Taylor [2] proved that the list chromatic number of the complete bipartite graph $K_{m,m}$ with m vertices in any part grows as binary logarithm of m:

$$\chi_l(K_{m,m}) = (1 + o(1)) \log_2(m) \text{ as } m \to \infty. \tag{1}$$

The above result was generalized in two different ways. On the first direction the researchers investigated the list chromatic number of K_{m*r}, the complete r-partite graph with equal size of parts m. Alon [3] showed that $\chi_l(K_{m*r}) = \Theta(r \ln(m))$. The asymptotics was obtained by Krivelevich and Gazit [4]: for fixed $r \geqslant 2$,

$$\chi_l(K_{m*r}) = (1 + o(1)) \log_{\frac{r}{r-1}}(m) \text{ as } m \to \infty. \tag{2}$$

In [5] Shabanov showed that the same asymptotic representation holds when $\ln(r) = o(\ln(m))$.

The second direction deals with estimating the list chromatic number of the complete r-uniform hypergraphs. Let $H_{m\times r}$ denote the the complete r-partite r-uniform hypergraph with m vertices in every part. Haxell and Verstraëte [6] proved that for fixed $r \geqslant 3$,

$$\chi_l(H_{m\times r}) = (1 + o(1)) \log_r(m) \text{ as } m \to \infty. \tag{3}$$

In the current work we give a parametric extension of the results (1)-(3). Let $H(m, r, k)$ denote the complete r-partite k-uniform hypergraph with m vertices in every part, in which any edge takes exactly one vertex from some $k \leqslant r$ parts. Formally, $H(m, r, k) = (V, E)$, where

- $V = V_1 \sqcup \ldots \sqcup V_r$ is the union of r disjoint vertex subsets, each of size m, $|V_i| = m$, $i = 1, \ldots, r$;
- E consists of all k-subsets of V which have at most one vertex in any part V_i, i.e.

$$E = \left\{ A \in \binom{V}{k} : |A \cap V_i| \leqslant 1 \text{ for any } i = 1, \ldots, r \right\}.$$

Clearly, $H(m, r, 2) = K_{m*r}$ and $H(m, r, r) = H_{m\times r}$. The main result of the paper provides the asymptotic behavior for the list chromatic number of $H(m, r, k)$.

Theorem 1. *For fixed* $2 \leqslant k \leqslant r$,

$$\chi_l(H(m, r, k)) = (1 + o(1)) \log_{\frac{r}{r-k+1}}(m) \text{ as } m \to \infty.$$

The same asymptotic representation holds for any functions $r = r(m)$, $k = k(m)$, *such that* $\ln r = o(\ln m)$.

In the next section we discuss the connection of the list chromatic numbers of the complete multi-partite hypergraphs with the extremal property B-type problems.

2 Extremal property B–type problems

The close connection of the list colorings of complete multi-partite graphs with the classical property B problem was realized by Erdős, Rubin and Taylor in [2]. Recall that in the property B problem we are asked to find the extremal value $m(n)$ which is equal to the minimum possible number of edges in an n-uniform hypergraph with chromatic number greater than 2. The quantitative relation between $\chi_l(K_{m,m})$ and $m(n)$ is the following.

Claim 1. *Suppose that $n, m \geqslant 2$ are integers.*

1. *If $2m < m(n)$ then $\chi_l(K_{m,m}) \leqslant n$.*
2. *If $m \geqslant m(n)$ then $\chi_l(K_{m,m}) > n$.*

These inequalities together with the known bounds for $m(n)$ (see [7],[8]),

$$m(n) = \Omega\left(\sqrt{\frac{n}{\ln n}} 2^n\right),\ m(n) = O\left(n^2 2^n\right),$$

provide the asymptotics for $\chi_l(K_{m,m})$.

The same logic was used in [5] for investigating $\chi_l(K_{m*r})$. In this case we have to consider the extremal problem for another type of colorings, which are called panchromatic. A vertex coloring of the hypergraph $H = (V, E)$ with r colors is said to be *panchromatic* if under this coloring every edge of E meets every of r colors, i.e. formally, $f : V \to \{1, \ldots, r\}$ is such that, for every edge $A \in E$,

$$|\{f(v) : v \in A\}| = r.$$

Let $p(n, r)$ denote the minimum possible number of edges in a n-uniform hypergraph that does not admit a panchromatic coloring with r colors. The fact that $p(n, r)$ plays the same role for $\chi_l(K_{m*r})$ as $m(n)$ for $\chi_l(K_{m,m})$, was proved by Kostochka in [9].

Claim 2. *Suppose that $n, m, r \geqslant 2$ are integers.*

1. *If $rm < p(n, r)$ then $\chi_l(K_{m*r}) \leqslant n$.*
2. *If $m \geqslant p(n, r)$ then $\chi_l(K_{m*r}) > n$.*

Another generalization of the property B problem was used by Haxell and Verstraëte in [6] to obtain the asymptotics for $\chi_l(H_{m\times r})$. Let $m(n, r)$ denote the minimum possible number of edges in an n-uniform hypergraph with chromatic number greater than r. This extremal value is also well-known and was studied for decades. The reader is referred to the survey [10] for the background of the problem. The best lower bounds for $m(n, r)$ were recently obtained by Cherkashin and Kozik [11] for $r < n$ and by Akolzin and Shabanov [12] for $r \geqslant n$. It is known that

$$m(n, r) = \Omega\left(\left(\frac{n}{\ln n}\right)^{\frac{r-1}{r}} r^{n-1}\right),\ m(n, r) = O\left(n^2 r^n \ln(r)\right).$$

The mentioned bounds together with the claim proved by Haxell and Verstraëte provide the asymptotics for $\chi_l(H_{m\times r})$.

Claim 3. *Suppose that $n, m, r \geqslant 2$ are integers.*

1. *If $rm < m(n,r)$ then $\chi_l(H_{m\times r}) \leqslant n$.*
2. *If $m \geqslant m(n,r)$ then $\chi_l(H_{m\times r}) > n$.*

In the current we work prove the extension of Claims 1-3 which then will be used for obtaining the asymptotics of $\chi_l(H(m,r,k))$. However, first we need to clarify what extremal value will help us. Since the hypergraph $H(m,r,k)$ "is between" $H_{m\times r}$ and K_{m*r}, it is natural to use the extremal value for colorings which "are between" the usual proper colorings ($m(n,r)$, every edge meets at least two colors) and panchromatic colors ($p(n,r)$, every edge meets all r colors). However, we were not able to deal with the property *every edge meets at least s colors*. This obstacle led us to a wider property concerning multiple coverings by independent sets.

Recall that a vertex subset $W \subset V$ is said to be *independent* in a hypergraph $H = (V,E)$ if W does not completely contain any edge of H, i.e. for every edge $A \in E$, $A \subsetneq W$. Let $[r] = \{1,\dots,r\}$. A mapping $f : V \to \binom{[r]}{s}$ is called an *s-covering* by r sets, i.e. we assign s different colors to any vertex of H. Finally, f is called an *s-covering by r independent sets* if for every $i = 1,\dots,r$, a vertex subset

$$V_i = \{v \in V : i \in f(v)\}$$

is an independent set in H.

Let us understand that the parametric family of properties "*a hypergraph admits an s-covering by r independent sets*", $s = 1,\dots,r-1$, contains the existence of a proper r-coloring and also the existence of a panchromatic r-coloring.

- Clearly, a 1-covering by r independent sets is just a proper coloring with r colors.
- If we have an $(r-1)$-covering f by r independent sets for $H = (V,E)$ then a coloring $f'(v) = [r] \setminus f(v)$ is panchromatic. Indeed, suppose f' contains an edge A without color j. Then for every vertex $v \in A$, we have $j \in f(v)$, hence, $V_j = \{v \in V : j \in f(v)\}$ is not independent. A contradiction. From the other hand, if f' is a panchromatic coloring with r colors for $H = (V,E)$ then $f(v) = [r] \setminus f'(v)$, $v \in V$, is an $(r-1)$-covering by r independent sets since for every $i = 1,\dots,r$, every edge has a vertex not containing in $V_i = \{v \in V : i \in f(v)\}$.

Let $c(n,r,s)$ denote the minimum possible of edges in an n-uniform hypergraph that does not admit an s-covering by r independent sets. The above discussion implies that $m(n,r) = c(n,r,1)$ and $p(n,r) = c(n,r,r-1)$. The proof of Theorem 1 relies on the following lemma which generalizes Claims 1-3.

Lemma 1. *Suppose that $n, m, r \geqslant 2$, $2 \leqslant k \leqslant r$ are integers.*

1. *If $rm < c(n,r,r-k+1)$ then $\chi_l(H(m,r,k)) \leqslant n$.*
2. *If $m \geqslant c(n,r,r-k+1)$ then $\chi_l(H(m,r,k)) > n$.*

The remaining part of the paper is organized as follows. In Section 3 we will prove Lemma 1. In Section 4 the bounds for the extremal value $c(n,r,s)$ will be deduced. In the final section we will prove Theorem 1.

3 Proof of Lemma 1

We follow the ideas from [6] and [9].

1) We have to show that $\chi_l(H(m,r,k)) \leqslant n$. Let $W = W_1 \sqcup \ldots \sqcup W_r$ be a vertex set of $H(m,r,k)$. Let $L = (L(w), w \in W)$ be an arbitrary n-uniform list assignment, $|L(w)| = n$ for any $w \in W$. Let us denote $C = \cup_{w \in W} L(w)$ and consider an n-uniform hypergraph $H' = (C, L)$. Since $|L| = |W| = rm < c(n, r, r-k+1)$ the hypergraph H' admits an $(r-k+1)$-covering f by r independent sets. Let us fix any such covering $f : C \to \binom{[r]}{r-k+1}$ and for every $i \in [r]$ let us denote $C_i = \{c \in C : i \in f(c)\}$.

Note that for any $i = 1, \ldots, r$ and any $w \in W_i$, the set $L(w)$ is not completely covered by C_i (since C_i is independent). Let us take any $c \in L(w)$ such that $c \notin C_i$ and set $f'(w) = c$. We will show that the obtained coloring $f' : W \to C$ is a proper coloring for $H(m,r,k)$ from the lists L. Suppose that there is a monochromatic edge $A \in E(H(n,r,k))$ of some color c' under this coloring. Assume that $A = \{w_1, \ldots, w_k\}$, $w_j \in W_{a_j}$ for some $1 \leqslant a_1 < \ldots < a_k \leqslant r$. Our construction implies that c' does not belong to any of the sets $C_{a_1}, \ldots, C_{a_k}$, but f is an $(r-k+1)$-covering, so c' should be covered by exactly $(r-k+1)$ sets among $C_1, \ldots, C_r$. A contradiction.

2) We have to show that $\chi_l(H(m,r,k)) > n$, i.e. there exists an n-uniform list assignment without a proper coloring from the lists. Since $m \geqslant c(n,r,r-k+1)$ there exists an n-uniform hypergraph $H' = (C, F)$ with m edges that does not admit an $(r-k+1)$-covering by r independent sets. Suppose that $F = \{F_1, \ldots, F_m\}$ and construct the following list assignment L for the vertices of $H(m,r,k)$.

Let $W = W_1 \sqcup \ldots \sqcup W_r$ be a vertex set of $H(m,r,k)$. If $W_i = \{w_{ij}, j = 1, \ldots, m\}$ then we set

$$L(w_{ij}) = F_j,\ j = 1, \ldots, m.$$

Suppose that there is a proper coloring $f' : W \to C$ from the lists L, i.e. $f'(w) \in L(w)$ for any $w \in W$. Then any $c \in C$ can be used on at most $k-1$ parts among $W_1, \ldots, W_r$. Let us define

$$f(c) = \{i \in [r] : f'(w) \neq c \text{ for any } w \in W_i\}, c \in C.$$

So, $|f(c)| \geqslant r - k + 1$ for any $c \in C$. We will show that every $C_i = \{c \in C : i \in f(c)\}$, $i = 1, \ldots, r$, is an independent set in H'. Indeed, every edge $F_j \in F$ is a list for w_{ij}, so there is $a \in F_j$ such that $f'(w_{ij}) = a$. Thus, $i \notin f(a)$ and $F_j \subsetneq C_i$. From f we can immediately construct an $(r-k+1)$-covering by r independent sets (it is enough to cut the sizes of $f(v)$ to $(r-k+1)$ by removing arbitrary vertices). A contradiction. Lemma 1 is proved.

4 Bounds for $c(n,r,s)$

In this section we will estimate the value $c(n,r,s)$ by using the probabilistic argument. Lemma2 provides the lower bound.

Lemma 2. *For any $n \geqslant 3$, $r \geqslant 2$, $1 \leqslant s \leqslant r-1$,*

$$c(n,r,s) \geqslant \frac{r^{n-1}}{s^n}. \tag{4}$$

Proof. Let $H = (V, E)$ be an n-uniform hypergraph with $|E| < \frac{r^{n-1}}{s^n}$. We have to show that there exists an s-covering by r independent sets. Consider a random mapping $f : V \to \binom{[r]}{s}$ with uniform distribution and let $C_i = \{v \in V : i \in f(v)\}$. Then for every edge A,

$$\mathsf{Pr}(A \subset C_i) = \left(\frac{\binom{r-1}{s-1}}{\binom{r}{s}}\right)^n = \left(\frac{s}{r}\right)^n.$$

Thus,

$$\mathsf{Pr}(\exists A \in E, \exists i \in [r] : A \subset C_i) \leqslant \sum_{A \in E} \sum_{i=1}^{r} \mathsf{Pr}(A \subset C_i) \leqslant |E| r \left(\frac{s}{r}\right)^n < 1.$$

Consequently, with positive probability every C_i is an independent set. This proves the existence of an s-covering by r independent sets.

□

Lemma 3 gives the upper bound.

Lemma 3. *For any $n > r \geqslant 2$, $1 \leqslant s \leqslant r - 1$,*

$$c(n, r, s) \leqslant \frac{e}{2} n^2 \left(\frac{r}{s}\right)^n \ln \binom{r}{s} \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{s}{r}\right)\right). \tag{5}$$

Proof. We have to show that there exists an n-uniform hypergraph $H = (V, E)$ that does not admit an s-covering by r independent sets and whose number of edges does not exceed the value in the right-hand side of (5).

Let V be a set of $v = \lceil (r/2s) n^2 \rceil$ vertices. Let $f : V \to \binom{[r]}{s}$ be an arbitrary s-covering of V and set $V_i = V_i(f) = \{v \in V : i \in f(v)\}$. Consider a random n-subset S taken from V with uniform distribution. The inclusion–exclusion principle implies

$$\mathsf{Pr}(\exists i : S \subset V_i) = \frac{1}{\binom{v}{n}} \left(\sum_{a=1}^{s} \sum_{1 \leqslant i_1 < \ldots < i_a \leqslant r} (-1)^{a-1} \binom{|V_{i_1} \cap \ldots \cap V_{i_a}|}{n} \right).$$

We need to estimate the sum in the right-hand side of the above equality. Let us denote it by $q(f)$. We will show that for any f,

$$q(f) = \sum_{a=1}^{s} \sum_{1 \leqslant i_1 < \ldots < i_a \leqslant r} (-1)^{a-1} \binom{|V_{i_1} \cap \ldots \cap V_{i_a}|}{n} \geqslant \frac{r}{s} \binom{\frac{vs}{r}}{n}. \tag{6}$$

The proof of the inequality (6) will be given in Claim 4 after the proof of the lemma.

Consider independent random n-subsets $S_1, \ldots, S_t$ taken from V with uniform distribution and construct a random hypergraph $H = (V, E)$ with $E = \{S_1, \ldots, S_t\}$. The bound (6) implies

$$\mathsf{Pr}(\exists f : \text{all } V_i(f) \text{ are independent in } H) \leqslant \sum_f \mathsf{Pr}(\text{all } V_i(f) \text{ are independent in } H) =$$

$$\sum_f \left(1 - \frac{q(f)}{\binom{v}{n}}\right)^t \leqslant \binom{r}{s}^v \left(1 - \frac{r}{s} \frac{\binom{\frac{vs}{r}}{n}}{\binom{v}{n}}\right)^t < \binom{r}{s}^v e^{-\frac{r}{s} pt},$$

where $p = p(v,r,s) = \frac{\binom{\frac{vs}{r}}{n}}{\binom{v}{n}}$. Since for $a = O(\sqrt{b})$,

$$\frac{(b)_a}{b^a} = \frac{(b)(b-1)\dots(b-a+1)}{b^a} = \left(1 + O\left(\frac{a}{b}\right)\right) e^{-\frac{a^2}{2b}}$$

and $v = \lceil (r/2s)n^2 \rceil$, we obtain

$$p = \frac{\binom{\frac{vs}{r}}{n}}{\binom{v}{n}} = \left(\frac{s}{r}\right)^n \left(1 + O\left(\frac{nr}{vs}\right)\right) e^{-\frac{n^2}{2}\left(\frac{r}{sv} - \frac{1}{v}\right)} =$$
$$\left(\frac{s}{r}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right) e^{-1+\frac{s}{r}} = e^{-1}\left(\frac{s}{r}\right)^n \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{s}{r}\right)\right).$$

The probability that there exists an s-covering by r independent sets for the random hypergraph H does not exceed $\exp\left(v \ln \binom{r}{s} - \frac{r}{s} pt\right)$. Hence, for

$$t = \left\lceil \frac{v \ln \binom{r}{s}}{\frac{r}{s} p} \right\rceil + 1 = \frac{e}{2} n^2 \left(\frac{r}{s}\right)^n \ln \binom{r}{s} \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{s}{r}\right)\right),$$

this probability is strictly less than 1. Finally, we established that with positive probability H does not admit an s-covering by r independent sets and has at most t edges. Thus, $c(n,r,s) \leqslant t$. Lemma 3 is proved.

□

It remains to deduce the technical inequality (6).

Claim 4. *For any s-covering $f : V \to \binom{[r]}{s}$ of V*

$$\sum_{j=1}^{s} \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant r} (-1)^{j-1} \binom{|V_{i_1} \cap \ldots \cap V_{i_j}|}{n} \geqslant \frac{r}{s} \binom{\frac{vs}{r}}{n},$$

where $V_i = V_i(f) = \{v \in V : i \in f(v)\}$.

Proof. Let us denote

$$A_j = \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant r} \binom{|V_{i_1} \cap \ldots \cap V_{i_j}|}{n}.$$

We want to estimate $T = \sum_{j=1}^{s} (-1)^{j-1} A_j$ and prove that $T \geqslant \frac{1}{s} A_1$. This will be sufficient to establish the claim since $\sum_{i=1}^{r} |V_i| = sv$ (recall that f is an s-covering), and, by the convexity, we have

$$T \geqslant \frac{1}{s} A_1 = \frac{1}{s} \sum_{i=1}^{r} \binom{|V_i|}{n} \geqslant \frac{r}{s} \binom{\frac{vs}{r}}{n}.$$

First, we will need some relations between the values A_j. Let us fix arbitrary $j < s$ and $1 \leqslant i_1 < \ldots < i_j \leqslant r$, denote $B = V_{i_1} \cap \ldots \cap V_{i_j}$. The inclusion–exclusion principle implies

$$\binom{|B|}{n} \geqslant \sum_{t=1}^{s-j} (-1)^{t-1} \sum_{\substack{1 \leqslant y_1 < \ldots < y_t \leqslant r: \\ y_1, \ldots, y_t \notin \{i_1, \ldots, i_j\}}} \binom{|B \cap V_{y_1} \cap \ldots \cap V_{y_t}|}{n}.$$

Summation of all the above inequalities over $1 \leqslant i_1 < \ldots < i_j \leqslant r$ leads to the following relation between A_j and $A_{j+1}, \ldots, A_s$:

$$A_j + \sum_{t=j+1}^{s} (-1)^{t-j} \binom{t}{j} A_t = \sum_{t=j}^{s} (-1)^{t-j} \binom{t}{j} A_t \geqslant 0. \tag{7}$$

Define the upper triangular matrix $D = \|d_{jt}\|_{j,t=1}^{s}$ where

$$d_{jt} = \begin{cases} (-1)^{t-j}\binom{t}{j}, & \text{if } t \geqslant j; \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Hence, the relation (7) implies that for any $j = 1, \ldots, s-1$, $\sum_{t=1}^{s} d_{jt} A_t \geqslant 0$. Note that d_{jj} equals 1 for any $j = 1, \ldots, s$. Let us make the following transformations with the first line of matrix D.

1. First, we add the second line multiplied by some $b_2 > 0$ from the first to make coefficient at A_2 equal to zero.
2. Let $D^{(2)}$ denote the obtained matrix.
3. Then for the first line of $D^{(2)}$, do the same: multiple the third line by some $b_3 > 0$ to make coefficient at A_3 equal to zero. The obtained matrix is denoted by $D^{(3)}$.
4. After $s-2$ steps of the procedure we get $(s-2)$ matrices $D^{(2)}, \ldots, D^{(s-1)}$ such that for every $i = 2, \ldots, s-1$,
   - $\sum_{t=1}^{s} d_{1t}^{(i)} A_t \geqslant 0$;
   - $d_{11}^{(i)} = 1$, $d_{1t}^{(i)} = 0$ for $t = 2, \ldots, i$.

Let us prove that b_i is always positive. In fact, we will show that $b_i = i$. Use the induction argument over i. If $i = 2$ then $d_{12} = -2$ and $d_{22} = 1$, so $b_2 = 2$. Assume that $b_i = i$ for any $i < m$. We need to find $d_{1m}^{(m-1)}$ for obtaining b_m. Due to the construction,

$$d_{1m}^{(m-1)} = d_{1m} + \sum_{i=2}^{m-1} b_i d_{im} = |(8)| = (-1)^{m-1}\binom{m}{1} + \sum_{i=2}^{m-1} i(-1)^{m-i}\binom{m}{i} =$$
$$(-1)^{m-1} m \left(1 + \sum_{i=2}^{m-1} (-1)^{i-1}\binom{m-1}{i-1}\right) = (-1)^{m-1} m \left(0 - (-1)^{m-1}\right) = -m.$$

Thus, $b_m = -d_{1m}^{(m-1)}/d_{mm} = m$.

The discussed transformations of D provide another series of inequalities for A_i. The relation $\sum_{t=1}^{s} d_{1t}^{(i-1)} A_t \geqslant 0$ can be rewritten as follows: since $d_{11}^{(i-1)} = 1$, $d_{1t}^{(i-1)} = 0$, $t = 2, \ldots, i-1$, and $d_{1i}^{(i-1)} = -i$, we obtain

$$A_1 - iA_i + \sum_{t=i+1}^{s} d_{1t}^{(i-1)} A_t \geqslant 0$$

or

$$A_i \leqslant \frac{1}{i}A_1 + \frac{1}{i}\sum_{t=i+1}^{s} d_{1t}^{(i-1)} A_t. \tag{9}$$

Now we are ready to estimate $T = \sum_{j=1}^{s}(-1)^{j-1}A_j$. First, we apply the inequality (9) to A_2:

$$T = A_1 - A_2 + A_3 = \ldots + (-1)^{s-1}A_s \geqslant A_1 - \frac{1}{2}\left(A_1 + \sum_{t=3}^{s} d_{1t}^{(1)} A_t\right) + \sum_{j=3}^{s}(-1)^{j-1}A_j = T_2.$$

The expression T_2 can be written as a linear combination of A_j for $j = 1, 3, \ldots, s$:

$$T_2 = \frac{1}{2}A_1 + \sum_{j=3}^{s} u_{2j}A_j.$$

If $u_{23} < 0$ then we again can apply (9) to A_3 and obtain $T_3 \leqslant T_2$, where T_3 will be a linear combination of A_j for $j = 1, 4, \ldots, s$. Suppose $T_i = u_{i1}A_1 + \sum_{j=i+1}^{s} u_{ij}A_s$ is obtained with $u_{i,i+1} < 0$ and $T_i \leqslant T_{i-1}$. Then we apply (9) to A_{i+1} and obtain T_{i+1}. Finally, T_s will be equal to $u_{s1}A_1$ for some $u_{s1} > 0$.

Let us show by the induction that for any $i = 2, \ldots, s-1$, $u_{1i} = -u_{i,i+1} = \frac{1}{i}$. For $i = 2$, we have already seen it. Assume that $u_{1i} = -u_{i,i+1} = \frac{1}{i}$ for $i < m$. Let us calculate the coefficient $u_{m,m+1}$ at A_{m+1} in T_m. It is equal to the initial coefficient $(-1)^m$ minus the sum of coefficients following from the applications of (9) for $A_2, \ldots, A_m$. If we apply (9) to A_i then we subtract $(1/i)d_{1,m+1}^{(i-1)}$ multiplied by $u_{i-1,i}$. By the induction $u_{i-1,i} = 1/(i-1)$, thus,

$$\begin{aligned}
u_{m,m+1} =& (-1)^m - \sum_{i=2}^{m} d_{1,m+1}^{(i-1)} \frac{1}{i(i-1)} = (-1)^m - \sum_{i=2}^{m}\left(d_{1,m+1} + \sum_{t=2}^{i-1} b_t d_{t,m+1}\right)\frac{1}{i(i-1)} = \\
& (-1)^m - \sum_{i=2}^{m}\sum_{t=1}^{i-1} t d_{t,m+1}\frac{1}{i(i-1)} = (-1)^m - \sum_{i=2}^{m}\sum_{t=1}^{i-1} t(-1)^{m+1-t}\binom{m+1}{t}\frac{1}{i(i-1)} = \\
& (-1)^m - \sum_{t=1}^{m-1} t(-1)^{m+1-t}\binom{m+1}{t}\sum_{i=t+1}^{m}\frac{1}{i(i-1)} = \\
& (-1)^m - \sum_{t=1}^{m-1} t(-1)^{m+1-t}\binom{m+1}{t}\left(\frac{1}{t} - \frac{1}{m}\right) = \\
& (-1)^m - \sum_{t=1}^{m-1}(-1)^{m+1-t}\binom{m+1}{t} + \frac{m+1}{m}\sum_{t=1}^{m-1}(-1)^{m+1-t}\binom{m}{t-1} = \\
& (-1)^m - (0 - (-1)^{m+1} - 1 + (m+1)) + \frac{m+1}{m}(0 - 1 + m) = \\
& 1 - \frac{m+1}{m} = -\frac{1}{m}.
\end{aligned}$$

The coefficient $u_{1,m+1}$ at A_1 is equal to

$$u_{1,m+1} = u_{1,m} - \frac{1}{m} u_{m-1,m} = \frac{1}{m-1} - \frac{1}{m(m-1)} = \frac{1}{m}.$$

Let us finish the proof. We have shown that $T \geqslant T_s = u_{s1} A_1 = A_1/s$. Claim 4 is proved. □

5 Proof of Theorem 1

Now we will deduce the main result concerning the asymptotics of $\chi_l(H(m,r,k))$. Let us denote $x = \chi_l(H(m,r,k))$. Lemma 1 implies

$$c(x-1, r, r-k+1) \leqslant m \text{ and } c(x, r, r-k+1) > mr.$$

By using bounds (4) and (5), we obtain

$$(x-2) \ln \left(\frac{r}{r-k+1} \right) - \ln(r-k+1) \leqslant \ln m; \tag{10}$$

$$\ln m + \ln r < x \ln \left(\frac{r}{r-k+1} \right) + 2 \ln x + \ln \ln \left(\frac{r}{r-k+1} \right) + O(1). \tag{11}$$

The condition of the theorem states that $\ln r = o(\ln m)$ as $m \to \infty$. Hence, the inequality (10) implies

$$\limsup_{m\to\infty} \frac{x \ln\left(\frac{r}{r-k+1}\right)}{\ln m} \leq 1 + \lim_{m\to\infty} \frac{3\ln r}{\ln m} = 1 \tag{12}$$

Moreover, it follows from (10) that $\ln x = O(\ln \ln m) = o(\ln m)$. Thus, by using (11), we have

$$\liminf_{m\to\infty} \frac{x \ln\left(\frac{r}{r-k+1}\right)}{\ln m} \geq 1 - \lim_{m\to\infty} \frac{O(\ln r + \ln x)}{\ln m} = 1. \tag{13}$$

Relations (12) and (13) provide the asymptotics for the list chromatic number of $\chi_l(H(m,r,k))$:

$$\lim_{m\to\infty} \frac{\chi_l(H(m,r,k)) \ln\left(\frac{r}{r-k+1}\right)}{\ln m} = \lim_{m\to\infty} \frac{\chi_l(H(m,r,k))}{\log_{\frac{r}{r-k+1}} m} = 1.$$

Theorem 1 is proved.

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