On the list chromatic number of the complete multi-partite hypergraphs and multiple coverings by independent sets

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Abstract

The paper deals with list colorings of hypergraphs. Consider the hypergraph $H(m, r, k)$, the complete $r$-partite $k$-uniform hypergraph with parts of equal size $m$, in which any edge takes exactly one vertex from some $k < r$ parts. Using the results concerning multiple coverings by independent sets we prove that for fixed $k$ and $r$, the list chromatic number of $H(m, r, k)$ is equal to $(1 + o(1)) \log_{\frac{r}{r-k+1}}(m)$ as $m \to \infty$.

Keywords: list colorings, colorings of hypergraphs, independent sets.

1 Introduction and the background of the problem

The work is devoted to the list colorings of uniform hypergraphs. Let us recall some definitions.

Let $H = (V, E)$ be a hypergraph. A vertex coloring $f$ is a mapping from the vertex set $V$ to some set of colors $C$. A coloring is called proper for $H$ if there is no monochromatic edges in $E$ under it, i.e. formally, for every edge $A \in E$,

$$|\{f(v) : v \in A\}| > 1.$$ 

The chromatic number of $H$, $\chi(H)$, is the minimum $s$ such that there is proper coloring for $H$ with $s$ colors.

The extension of the chromatic number is the notion of the list chromatic number. A hypergraph $H = (V, E)$ is said to be $s$-choosable if for every family of sets $L = \{L(v) : v \in V\}$ ($L$ is called the list assignment), such that $|L(v)| = s$ for all $v \in V$ ($s$-uniform list assignment), there is a proper coloring from the lists, i.e. for every $v \in V$, we should use a color from

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The list chromatic number of $H$, denoted by $\chi_l(H)$, is the minimum $s$ such that $H$ is $s$-choosable.

The study of list colorings of graphs and hypergraphs was initiated by Vizing [1] and by Erdős, Rubin and Taylor [2]. One of the first interesting results concerning the list chromatic number is that it can be much larger than the usual chromatic number. In particular, Erdős, Rubin and Taylor [2] proved that the list chromatic number of the complete bipartite graph $K_{m,m}$ with $m$ vertices in any part grows as binary logarithm of $m$:

$$\chi_l(K_{m,m}) = (1 + o(1)) \log_2(m) \text{ as } m \to \infty. \quad (1)$$

The above result was generalized in two different ways. On the first direction the researchers investigated the list chromatic number of $K_{m^r}$, the complete $r$-partite graph with equal size of parts $m$. Alon [3] showed that $\chi_l(K_{m^r}) = \Theta(r \ln(m))$. The asymptotics was obtained by Krivelevich and Gazit [4]: for fixed $r \geq 2$,

$$\chi_l(K_{m^r}) = (1 + o(1)) \log_{r^{r-1}}(m) \text{ as } m \to \infty. \quad (2)$$

In [5] Shabanov showed that the same asymptotic representation holds when $\ln(r) = o(\ln(m))$.

The second direction deals with estimating the list chromatic number of the complete $r$-uniform hypergraphs. Let $H_{m^r}$ denote the the complete $r$-partite $r$-uniform hypergraph with $m$ vertices in every part. Haxell and Verstraëte [6] proved that for fixed $r \geq 3$,

$$\chi_l(H_{m^r}) = (1 + o(1)) \log_r(m) \text{ as } m \to \infty. \quad (3)$$

In the current work we give a parametric extension of the results (1)-(3). Let $H(m,r,k)$ denote the complete $r$-partite $k$-uniform hypergraph with $m$ vertices in every part, in which any edge takes exactly one vertex from some $k \leq r$ parts. Formally, $H(m,r,k) = (V,E)$, where

- $V = V_1 \sqcup \ldots \sqcup V_r$ is the union of $r$ disjoint vertex subsets, each of size $m$, $|V_i| = m$, $i = 1, \ldots, r$;
- $E$ consists of all $k$-subsets of $V$ which have at most one vertex in any part $V_i$, i.e.

$$E = \left\{ A \in \binom{V}{k} : |A \cap V_i| \leq 1 \text{ for any } i = 1, \ldots, r \right\}.$$ 

Clearly, $H(m,r,2) = K_{m^r}$ and $H(m,r,r) = H_{m^r}$. The main result of the paper provides the asymptotic behavior for the list chromatic number of $H(m,r,k)$.

**Theorem 1.** For fixed $2 \leq k \leq r$,

$$\chi_l(H(m,r,k)) = (1 + o(1)) \log_{r^{r-1}}(m) \text{ as } m \to \infty.$$ 

The same asymptotic representation holds for any functions $r = r(m)$, $k = k(m)$, such that $\ln r = o(\ln m)$.

In the next section we discuss the connection of the list chromatic numbers of the complete multi-partite hypergraphs with the extremal property B-type problems.
2 Extremal property B–type problems

The close connection of the list colorings of complete multi-partite graphs with the classical property B problem was realized by Erdős, Rubin and Taylor in [2]. Recall that in the property B problem we are asked to find the extremal value \( m(n) \) which is equal to the minimum possible number of edges in an \( n \)-uniform hypergraph with chromatic number greater than 2. The quantitative relation between \( \chi_l(K_{m,m}) \) and \( m(n) \) is the following.

**Claim 1.** Suppose that \( n, m \geq 2 \) are integers.

1. If \( 2m < m(n) \) then \( \chi_l(K_{m,m}) \leq n \).
2. If \( m \geq m(n) \) then \( \chi_l(K_{m,m}) > n \).

These inequalities together with the known bounds for \( m(n) \) (see [7],[8]),

\[
m(n) = \Omega \left( \sqrt{n \ln n} \right), \quad m(n) = O \left( n^2 2^n \right),
\]

provide the asymptotics for \( \chi_l(K_{m,m}) \).

The same logic was used in [5] for investigating \( \chi_l(K_{m,r}) \). In this case we have to consider the extremal problem for another type of colorings, which are called panchromatic. A vertex coloring of the hypergraph \( H = (V,E) \) with \( r \) colors is said to be panchromatic if under this coloring every edge of \( E \) meets every of \( r \) colors, i.e. formally, \( f : V \to \{1, \ldots, r\} \) is such that, for every edge \( A \in E \),

\[
|\{f(v) : v \in A\}| = r.
\]

Let \( p(n, r) \) denote the minimum possible number of edges in a \( n \)-uniform hypergraph that does not admit a panchromatic coloring with \( r \) colors. The fact that \( p(n, r) \) plays the same role for \( \chi_l(K_{m,r}) \) as \( m(n) \) for \( \chi_l(K_{m,m}) \), was proved by Kostochka in [9].

**Claim 2.** Suppose that \( n, m, r \geq 2 \) are integers.

1. If \( rm < p(n, r) \) then \( \chi_l(K_{m,r}) \leq n \).
2. If \( m \geq p(n, r) \) then \( \chi_l(K_{m,r}) > n \).

Another generalization of the property B problem was used by Haxell and Verstraëte in [6] to obtain the asymptotics for \( \chi_l(H_{m,r}) \). Let \( m(n, r) \) denote the minimum possible number of edges in an \( n \)-uniform hypergraph with chromatic number greater than \( r \). This extremal value is also well-known and was studied for decades. The reader is referred to the survey [10] for the background of the problem. The best lower bounds for \( m(n, r) \) were recently obtained by Cherkashin and Kozik [11] for \( r < n \) and by Akolzin and Shabanov [12] for \( r \geq n \). It is known that

\[
m(n, r) = \Omega \left( \left( \frac{n}{\ln n} \right)^{\frac{r-1}{r}} r^{n-1} \right), \quad m(n, r) = O \left( n^2 r^n \ln r \right).
\]

The mentioned bounds together with the claim proved by Haxell and Verstraëte provide the asymptotics for \( \chi_l(H_{m,r}) \).
Claim 3. Suppose that $n, m, r \geq 2$ are integers.

1. If $rm < m(n, r)$ then $\chi_l(H_{m \times r}) \leq n$.
2. If $m \geq m(n, r)$ then $\chi_l(H_{m \times r}) > n$.

In the current we work prove the extension of Claims 1-3 which then will be used for obtaining the asymptotics of $\chi_l(H(m, r, k))$. However, first we need to clarify what extremal value will help us. Since the hypergraph $H(m, r, k)$ “is between” $H_{m \times r}$ and $K_{m \times r}$, it is natural to use the extremal value for colorings which “are between” the usual proper colorings $(m(n, r)$, every edge meets at least two colors) and panchromatic colors $(p(n, r)$, every edge meets all $r$ colors). However, we were not able to deal with the property every edge meets at least $s$ colors. This obstacle led us to a wider property concerning multiple coverings by independent sets.

Recall that a vertex subset $W \subset V$ is said to be independent in a hypergraph $H = (V, E)$ if $W$ does not completely contain any edge of $H$, i.e. for every edge $A \in E$, $A \not\subseteq W$. Let $[r] = \{1, \ldots, r\}$. A mapping $f : V \rightarrow \binom{[r]}{s}$ is called an $s$-covering by $r$ sets, i.e. we assign $s$ different colors to any vertex of $H$. Finally, $f$ is called an $s$-covering by $r$ independent sets if for every $i = 1, \ldots, r$, a vertex subset $V_i = \{v \in V : i \in f(v)\}$ is an independent set in $H$.

Let us understand that the parametric family of properties “a hypergraph admits an $s$-covering by $r$ independent sets”, $s = 1, \ldots, r - 1$, contains the existence of a proper $r$-coloring and also the existence of a panchromatic $r$-coloring.

- Clearly, a 1-covering by $r$ independent sets is just a proper coloring with $r$ colors.
- If we have an $(r - 1)$-covering $f$ by $r$ independent sets for $H = (V, E)$ then a coloring $f'(v) = [r] \setminus f(v)$ is panchromatic. Indeed, suppose $f'$ contains an edge $A$ without color $j$. Then for every vertex $v \in A$, we have $j \in f(v)$, hence, $V_j = \{v \in V : j \in f(v)\}$ is not independent. A contradiction. From the other hand, if $f'$ is a panchromatic coloring with $r$ colors for $H = (V, E)$ then $f(v) = [r] \setminus f'(v)$, $v \in V$, is an $(r - 1)$-covering by $r$ independent sets since for every $i = 1, \ldots, r$, every edge has a vertex not containing in $V_i = \{v \in V : i \in f'(v)\}$.

Let $c(n, r, s)$ denote the minimum possible of edges in an $n$-uniform hypergraph that does not admit an $s$-covering by $r$ independent sets. The above discussion implies that $m(n, r) = c(n, r, 1)$ and $p(n, r) = c(n, r, r - 1)$. The proof of Theorem 1 relies on the following lemma which generalizes Claims 1-3.

Lemma 1. Suppose that $n, m, r \geq 2$, $2 \leq k \leq r$ are integers.

1. If $rm < c(n, r, r - k + 1)$ then $\chi_l(H(m, r, k)) \leq n$.
2. If $m \geq c(n, r, r - k + 1)$ then $\chi_l(H(m, r, k)) > n$.

The remaining part of the paper is organized as follows. In Section 3 we will prove Lemma 1. In Section 4 the bounds for the extremal value $c(n, r, s)$ will be deduced. In the final section we will prove Theorem 1.
3 Proof of Lemma 1

We follow the ideas from [6] and [9].

1) We have to show that \( \chi_l(H(m, r, k)) \leq n \). Let \( W = W_1 \sqcup \ldots \sqcup W_r \) be a vertex set of \( H(m, r, k) \). Let \( L = (L(w), w \in W) \) be an arbitrary \( n \)-uniform list assignment, \( |L(w)| = n \) for any \( w \in W \). Let us denote \( C = \bigcup_{w \in W} L(w) \) and consider an \( n \)-uniform hypergraph \( H' = (C, L) \). Since \( |L| = |W| = rm < c(n, r, r - k + 1) \) the hypergraph \( H' \) admits an \((r - k + 1)\)-covering \( f \) by \( r \) independent sets. Let us fix any such covering \( f: C \to \binom{[r]}{r-k+1} \) and for every \( i \in [r] \) let us denote \( C_i = \{ c \in C : i \in f(c) \} \).

Note that for any \( i = 1, \ldots, r \) and any \( w \in W_i \), the set \( L(w) \) is not completely covered by \( C_i \) (since \( C_i \) is independent). Let us take any \( c \in L(w) \) such that \( c \notin C_i \) and set \( f'(w) = c \). We will show that the obtained coloring \( f': W \to C \) is a proper coloring for \( H(m, r, k) \) from the lists \( L \). Suppose that there is a monochromatic edge \( A \in E(H(n, r, k)) \) of some color \( c' \) under this coloring. Assume that \( A = \{w_1, \ldots, w_k\} \), \( w_j \in W_{a_j} \) for some \( 1 \leq a_1 < \ldots < a_k \leq r \). Our construction implies that \( c' \) does not belong to any of the sets \( C_{a_1}, \ldots, C_{a_k} \), but \( f \) is an \((r - k + 1)\)-covering, so \( c' \) should be covered by exactly \((r - k + 1)\) sets among \( C_1, \ldots, C_r \). A contradiction.

2) We have to show that \( \chi_l(H(m, r, k)) > n \), i.e. there exists an \( n \)-uniform list assignment without a proper coloring from the lists. Since \( m \geq c(n, r, r - k + 1) \) there exists an \( n \)-uniform hypergraph \( H' = (C, F) \) with \( m \) edges that does not admit an \((r - k + 1)\)-covering by \( r \) independent sets. Suppose that \( F = \{F_1, \ldots, F_m\} \) and construct the following list assignment \( L \) for the vertices of \( H(m, r, k) \).

Let \( W = W_1 \sqcup \ldots \sqcup W_r \) be a vertex set of \( H(m, r, k) \). If \( W_i = \{w_{ij}, j = 1, \ldots, m\} \) then we set

\[
L(w_{ij}) = F_j, \quad j = 1, \ldots, m.
\]

Suppose that there is a proper coloring \( f': W \to C \) from the lists \( L \), i.e. \( f'(w) \in L(w) \) for any \( w \in W \). Then any \( c \in C \) can be used on at most \( k - 1 \) parts among \( W_1, \ldots, W_r \). Let us define

\[
f(c) = \{ i \in [r] : f'(w) \neq c \text{ for any } w \in W_i \}, \quad c \in C.
\]

So, \( |f(c)| \geq r - k + 1 \) for any \( c \in C \). We will show that every \( C_i = \{ c \in C : i \in f(c) \} \), \( i = 1, \ldots, r \), is an independent set in \( H' \). Indeed, every edge \( F_j \in F \) is a list for \( w_{ij} \), so there is \( a \in F_j \) such that \( f'(w_{ij}) = a \). Thus, \( i \notin f(a) \) and \( F_j \subseteq C_i \). From \( f \) we can immediately construct an \((r - k + 1)\)-covering by \( r \) independent sets (it is enough to cut the sizes of \( f(v) \) to \((r - k + 1)\) by removing arbitrary vertices). A contradiction. Lemma 1 is proved.

4 Bounds for \( c(n, r, s) \)

In this section we will estimate the value \( c(n, r, s) \) by using the probabilistic argument. Lemma 2 provides the lower bound.

**Lemma 2.** For any \( n \geq 3, \ r \geq 2, \ 1 \leq s \leq r - 1 \),

\[
c(n, r, s) \geq \frac{r^{n-1}}{s^n}. \tag{4}
\]
Let \( H = (V, E) \) be an \( n \)-uniform hypergraph with \(|E| < \frac{n^{r-1}}{s^n} \). We have to show that there exists an \( s \)-covering by \( r \) independent sets. Consider a random mapping \( f : V \to \binom{[r]}{s} \) with uniform distribution and let \( C_i = \{ v \in V : i \in f(v) \} \). Then for every edge \( A \),

\[
\Pr(A \subset C_i) = \left( \frac{\binom{r-1}{s-1}}{\binom{r}{s}} \right)^n = \left( \frac{s}{r} \right)^n.
\]

Thus,

\[
\Pr(\exists A \in E, \exists i \in [r] : A \subset C_i) \leq \sum_{A \in E} \sum_{i=1}^r \Pr(A \subset C_i) \leq |E| r \left( \frac{s}{r} \right)^n < 1.
\]

Consequently, with positive probability every \( C_i \) is an independent set. This proves the existence of an \( s \)-covering by \( r \) independent sets.

Lemma 3 gives the upper bound.

**Lemma 3.** For any \( n > r \geq 2 \), \( 1 \leq s \leq r - 1 \),

\[
c(n, r, s) \leq \frac{e}{2} n^2 \left( \frac{r}{s} \right)^n \ln \left( \frac{r}{s} \right) \left( 1 + O \left( \frac{1}{n} \right) + O \left( \frac{s}{r} \right) \right).
\]

**Proof.** We have to show that there exists an \( n \)-uniform hypergraph \( H = (V, E) \) that does not admit an \( s \)-covering by \( r \) independent sets and whose number of edges does not exceed the value in the right-hand side of (5).

Let \( V \) be a set of \( v = [(r/2s)n^2] \) vertices. Let \( f : V \to \binom{[r]}{s} \) be an arbitrary \( s \)-covering of \( V \) and set \( V_i = V_i(f) = \{ v \in V : i \in f(v) \} \). Consider a random \( n \)-subset \( S \) taken from \( V \) with uniform distribution. The inclusion–exclusion principle implies

\[
\Pr(\exists i : S \subset V_i) = \frac{1}{\binom{v}{n}} \left( \sum_{a=1}^s \sum_{1 \leq i_1 < \ldots < i_a \leq r} (-1)^{a-1} \binom{|V_{i_1} \cap \ldots \cap V_{i_a}|}{n} \right).
\]

We need to estimate the sum in the right-hand side of the above equality. Let us denote it by \( q(f) \). We will show that for any \( f \),

\[
q(f) = \sum_{a=1}^s \sum_{1 \leq i_1 < \ldots < i_a \leq r} (-1)^{a-1} \binom{|V_{i_1} \cap \ldots \cap V_{i_a}|}{n} \geq r \left( \frac{us}{n} \right) \frac{r}{s} \frac{n}{r} \left( \frac{us}{n} \right) \left( \frac{r}{s} \right)
\]

The proof of the inequality (6) will be given in Claim 4 after the proof of the lemma.

Consider independent random \( n \)-subsets \( S_1, \ldots, S_t \) taken from \( V \) with uniform distribution and construct a random hypergraph \( H = (V, E) \) with \( E = \{ S_1, \ldots, S_t \} \). The bound (6) implies

\[
\Pr(\exists f : \text{all } V_i(f) \text{ are independent in } H) \leq \sum_f \Pr(\text{all } V_i(f) \text{ are independent in } H) = \sum_f \left( 1 - \frac{q(f)}{\binom{v}{n}} \right)^t \leq \left( \frac{r}{s} \right)^t \left( \frac{v}{n} \right)^t \left( \frac{r}{s} \right)^t \left( \frac{v}{n} \right)^t e^{-\frac{r}{s} n t},
\]

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Proof. We want to estimate $p = p(v, r, s) = \frac{{\binom{v}{s}}}{\binom{r}{s}}$. Since for $a = O(\sqrt{d})$,
\[
\frac{(b)_a}{b^a} = \frac{(b)(b-1) \ldots (b-a+1)}{b^a} = \left(1 + O\left(\frac{a}{b}\right)\right) e^{-\frac{a^2}{2b}}
\]
and $v = \left\lfloor (r/2s)n^2 \right\rfloor$, we obtain
\[
p = \frac{\binom{v}{n}}{\binom{r}{n}} = \left(\frac{s}{r}\right)^n \left(1 + O\left(\frac{nr}{vs}\right)\right) e^{-\frac{n^2}{2s} \left(\frac{s}{n} \cdot \frac{1}{r} - \frac{1}{s}\right)} = \left(\frac{s}{r}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right) e^{-1 + \frac{2}{s}} = \left(\frac{s}{r}\right)^n \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{s}{r}\right)\right).
\]

The probability that there exists an $s$-covering by $r$ independent sets for the random hypergraph $H$ does not exceed $\exp \left( v \ln \left(\frac{r}{s}\right) - \frac{s}{r} pt \right)$. Hence, for
\[
t = \left\lceil \frac{v \ln \left(\frac{r}{s}\right)}{\frac{s}{r} p} \right\rceil + 1 = \frac{1}{2} n^2 \left(\frac{r}{s}\right)^n \ln \left(\frac{r}{s}\right) \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{s}{r}\right)\right),
\]
this probability is strictly less than 1. Finally, we established that with positive probability $H$ does not admit an $s$-covering by $r$ independent sets and has at most $t$ edges. Thus, $c(n, r, s) \leq t$.

Lemma 3 is proved.

\[\square\]

It remains to deduce the technical inequality (6).

Claim 4. For any $s$-covering $f : V \to \binom{[r]}{s}$ of $V$
\[
\sum_{j=1}^{s} \sum_{1 \leq i_1 < \ldots < i_j \leq r} (-1)^{j-1} \left(\binom{|V_{i_1} \cap \ldots \cap V_{i_j}|}{n}\right) \geq \frac{r}{s} \left(\frac{v_a}{r}\right),
\]
where $V_i = V_i(f) = \{ v \in V : i \in f(v) \}$.

Proof. Let us denote
\[
A_j = \sum_{1 \leq i_1 < \ldots < i_j \leq r} \left(\binom{|V_{i_1} \cap \ldots \cap V_{i_j}|}{n}\right).
\]
We want to estimate $T = \sum_{j=1}^{s} (-1)^{j-1} A_j$ and prove that $T \geq \frac{1}{s} A_1$. This will be sufficient to establish the claim since $\sum_{i=1}^{r} |V_i| = sv$ (recall that $f$ is an $s$-covering), and, by the convexity, we have
\[
T \geq \frac{1}{s} A_1 = \frac{1}{s} \sum_{i=1}^{r} \left(\binom{|V_i|}{n}\right) \geq \frac{r}{s} \left(\frac{v_a}{r}\right).
\]

First, we will need some relations between the values $A_j$. Let us fix arbitrary $j < s$ and $1 \leq i_1 < \ldots < i_j \leq r$, denote $B = V_{i_1} \cap \ldots \cap V_{i_j}$. The inclusion–exclusion principle implies
\[
\binom{|B|}{n} \geq \sum_{t=1}^{s-j} (-1)^{t-1} \sum_{1 \leq y_1 < \ldots < y_t \leq r; y_1, \ldots, y_t \notin \{i_1, \ldots, i_j\}} \left(\binom{|B \cap V_{y_1} \cap \ldots \cap V_{y_t}|}{n}\right).
\]
Summation of all the above inequalities over $1 \leq i_1 < \ldots < i_j \leq r$ leads to the following relation between $A_j$ and $A_{j+1}, \ldots, A_s$:

$$A_j + \sum_{t=j+1}^s (-1)^{t-j} \binom{t}{j} A_t = \sum_{t=j}^s (-1)^{t-j} \binom{t}{j} A_t \geq 0. \quad (7)$$

Define the upper triangular matrix $D = \|d_{jt}\|_{j,t=1}^s$ where

$$d_{jt} = \begin{cases} (-1)^{t-j} \binom{t}{j}, & \text{if } t \geq j; \\ 0, & \text{otherwise}. \end{cases} \quad (8)$$

Hence, the relation (7) implies that for any $j = 1, \ldots, s$, $\sum_{t=1}^s d_{jt} A_t \geq 0$. Note that $d_{jj}$ equals 1 for any $j = 1, \ldots, s$. Let us make the following transformations with the first line of matrix $D$.

1. First, we add the second line multiplied by some $b_2 > 0$ from the first to make coefficient at $A_2$ equal to zero.

2. Let $D^{(2)}$ denote the obtained matrix.

3. Then for the first line of $D^{(2)}$, do the same: multiple the third line by some $b_3 > 0$ to make coefficient at $A_3$ equal to zero. The obtained matrix is denoted by $D^{(3)}$.

4. After $s - 2$ steps of the procedure we get $(s - 2)$ matrices $D^{(2)}, \ldots, D^{(s-1)}$ such that for every $i = 2, \ldots, s - 1$,

- $\sum_{t=1}^s d_{1t}^{(i)} A_t \geq 0$;
- $d_{11}^{(i)} = 1$, $d_{1t}^{(i)} = 0$ for $t = 2, \ldots, i$.

Let us prove that $b_i$ is always positive. In fact, we will show that $b_i = i$. Use the induction argument over $i$. If $i = 2$ then $d_{12} = -2$ and $d_{22} = 1$, so $b_2 = 2$. Assume that $b_i = i$ for any $i < m$. We need to find $d_{1m}^{(m-1)}$ for obtaining $b_m$. Due to the construction,

$$d_{1m}^{(m-1)} = d_{1m} + \sum_{i=2}^{m-1} b_i d_{im} = |(8)| = (-1)^{m-1} \binom{m}{1} + \sum_{i=2}^{m-1} i (-1)^{m-i} \binom{m}{i} =$$

$$(-1)^{m-1} m \left( 1 + \sum_{i=2}^{m-1} (-1)^{i-1} \binom{m-1}{i-1} \right) = (-1)^{m-1} m (0 - (-1)^{m-1}) = -m.$$

Thus, $b_m = -d_{1m}^{(m-1)}/d_{mm} = m$.

The discussed transformations of $D$ provide another series of inequalities for $A_i$. The relation $\sum_{t=1}^s d_{1t}^{(i-1)} A_t \geq 0$ can be rewritten as follows: since $d_{11}^{(i-1)} = 1$, $d_{1t}^{(i-1)} = 0$, $t = 2, \ldots, i - 1$, and $d_{1i}^{(i-1)} = -i$, we obtain

$$A_1 - i A_i + \sum_{t=i+1}^s d_{1t}^{(i-1)} A_t \geq 0$$
or

\[ A_i \leq \frac{1}{i} A_1 + \frac{1}{i} \sum_{t=i+1}^{s} d_{1t}^{(i-1)} A_t. \] (9)

Now we are ready to estimate \( T = \sum_{j=1}^{s} (-1)^{j-1} A_j \). First, we apply the inequality (9) to \( A_2 \):

\[ T = A_1 - A_2 + A_3 = \ldots + (-1)^{s-1} A_s \geq A_1 - \frac{1}{2} \left( A_1 + \sum_{t=3}^{s} d_{1t}^{(1)} A_t \right) + \sum_{j=3}^{s} (-1)^{j-1} A_j = T_2. \]

The expression \( T_2 \) can be written as a linear combination of \( A_j \) for \( j = 1, 3, \ldots, s \):

\[ T_2 = \frac{1}{2} A_1 + \sum_{j=3}^{s} u_{2j} A_j. \]

If \( u_{23} < 0 \) then we again can apply (9) to \( A_3 \) and obtain \( T_3 \leq T_2 \), where \( T_3 \) will be a linear combination of \( A_j \) for \( j = 1, 4, \ldots, s \). Suppose \( T_i = u_{i1} A_1 + \sum_{j=i+1}^{s} u_{ij} A_j \) is obtained with \( u_{i,i+1} < 0 \) and \( T_i \leq T_{i-1} \). Then we apply (9) to \( A_{i+1} \) and obtain \( T_{i+1} \). Finally, \( T_s \) will be equal to \( u_{s1} A_1 \) for some \( u_{s1} > 0 \).

Let us show by the induction that for any \( i = 2, \ldots, s-1, \) \( u_{1i} = -u_{i,i+1} = \frac{1}{i} \). For \( i = 2 \), we have already seen it. Assume that \( u_{1i} = -u_{i,i+1} = \frac{1}{i} \) for \( i < m \). Let us calculate the coefficient \( u_{m,m+1} \) at \( A_{m+1} \) in \( T_m \). It is equal to the initial coefficient \( (-1)^m \) minus the sum of coefficients following from the applications of (9) for \( A_2, \ldots, A_m \). If we apply (9) to \( A_i \) then we subtract \((1/i)d_{1,i+1}^{(i-1)}\) multiplied by \( u_{i-1,i} \). By the induction \( u_{i-1,i} = 1/(i-1) \), thus,

\[
\begin{align*}
\sum_{i=2}^{m} d_{1,i+1}^{(i-1)} \frac{1}{i(i-1)} &= (-1)^m \sum_{i=2}^{m} \left( d_{1,i+1}^{(1)} + \sum_{t=2}^{i-1} b_t d_{t,i+1}^{(1)} \right) \frac{1}{i(i-1)} = \\
(-1)^m - \sum_{i=2}^{m} \sum_{t=1}^{i-1} t d_{t,i+1}^{(1)} \frac{1}{i(i-1)} &= (-1)^m - \sum_{i=2}^{m} \sum_{t=1}^{i-1} t (-1)^{m-1-t} \binom{m+1}{t} \frac{1}{i(i-1)} = \\
(-1)^m - \sum_{t=1}^{m-1} t (-1)^{m+1-t} \binom{m+1}{t} \frac{1}{i(i-1)} &= \\
(-1)^m - \sum_{t=1}^{m-1} t (-1)^{m+1-t} \binom{m+1}{t} \left( \frac{1}{t} - \frac{1}{m} \right) &= \\
(-1)^m - \sum_{t=1}^{m-1} (-1)^{m+1-t} \binom{m+1}{t} + \frac{m+1}{m} \sum_{t=1}^{m-1} (-1)^{m+1-t} \binom{m}{t-1} &= \\
(-1)^m - (0 - (-1)^{m+1} - 1 + (m+1)) + \frac{m+1}{m} (0 - 1 + m) &= \\
1 - \frac{m+1}{m} = -\frac{1}{m}. 
\end{align*}
\]
The coefficient $u_{1,m+1}$ at $A_1$ is equal to

$$u_{1,m+1} = u_{1,m} - \frac{1}{m}u_{m-1,m} = \frac{1}{m-1} - \frac{1}{m(m-1)} = \frac{1}{m}.$$  

Let us finish the proof. We have shown that $T \geq T_s = u_{s1}A_1 = A_1/s$. Claim 4 is proved. □

5 Proof of Theorem 1

Now we will deduce the main result concerning the asymptotics of $\chi_l(H(m,r,k))$. Let us denote $x = \chi_l(H(m,r,k))$. Lemma 1 implies

$$c(x - 1, r, r - k + 1) \leq m \text{ and } c(x, r, r - k + 1) > mr.$$  

By using bounds (4) and (5), we obtain

$$(x - 2) \ln \left( \frac{r}{r - k + 1} \right) - \ln(r - k + 1) \leq \ln m; \quad (10)$$

$$\ln m + \ln r < x \ln \left( \frac{r}{r - k + 1} \right) + 2 \ln x + \ln \ln \left( \frac{r}{r - k + 1} \right) + O(1). \quad (11)$$

The condition of the theorem states that $\ln r = o(\ln m)$ as $m \to \infty$. Hence, the inequality (10) implies

$$\limsup_{m \to \infty} \frac{x \ln \left( \frac{r}{r - k + 1} \right)}{\ln m} \leq 1 + \lim_{m \to \infty} \frac{3 \ln r}{\ln m} = 1 \quad (12)$$

Moreover, it follows from (10) that $\ln x = O(\ln \ln m) = o(\ln m)$. Thus, by using (11), we have

$$\liminf_{m \to \infty} \frac{x \ln \left( \frac{r}{r - k + 1} \right)}{\ln m} \geq 1 - \lim_{m \to \infty} \frac{O(\ln r + \ln x)}{\ln m} = 1. \quad (13)$$

Relations (12) and (13) provide the asymptotics for the list chromatic number of $\chi_l(H(m,r,k))$:

$$\lim_{m \to \infty} \frac{\chi_l(H(m,r,k)) \ln \left( \frac{r}{r - k + 1} \right)}{\ln m} = \lim_{m \to \infty} \frac{\chi_l(H(m,r,k))}{\ln \left( \frac{r}{r - k + 1} \right) m} = 1.$$  

Theorem 1 is proved.

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References


