On the strong chromatic number of a random 3-uniform hypergraph

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Abstract

The paper deals with estimating the threshold for the strong r-colorability of a random 3-uniform hypergraph in the binomial model $H(n, 3, p)$. Recall that a vertex coloring is said to be strong for a hypergraph if every two vertices sharing a common edge are colored with distinct colors. It is known that the threshold corresponds to the sparse case, when the expected number of edges is a linear function of n, $p\binom{n}{k} = cn$, and $c > 0$. We establish the threshold as a bound on the parameter c up to an additive constant. In particular, by using the second moment method we prove that for large enough r and $c < \frac{r \ln r}{3} - \frac{5}{18} \ln r - \frac{1}{3} - r^{-1/6}$, the random hypergraph $H(n, 3, p)$ is strongly r-colorable with high probability and, vice versa, for $c > \frac{r \ln r}{3} - \frac{5}{18} \ln r + O(\ln r/r)$, it is not strongly r-colorable with high probability.

Keywords: random hypergraphs, strong colorings, second moment method.

1 Introduction

The paper deals with strong colorings of random hypergraphs. Let us start with recalling some definitions.

1.1 Definitions

Suppose $H = (V, E)$ is a hypergraph and $f : V \to \{1, \dots, r\}$ is a vertex coloring with r colors. Then f is said to be *strong* for the hypergraph H, if for every edge $A \in E$, all the vertices in A are colored with distinct colors, i.e.

$$|\{f(v) : v \in A\}| = |A|, \text{ for all } A \in E.$$

The minimum number of colors required for a strong coloring is called *the strong chromatic number* of the hypergraph H and is denoted by $\chi_s(H)$. Recall that the usual hypergraph

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chromatic number, $\chi(H)$, introduced by Erdős corresponds to *weak* colorings, i.e. colorings without monochromatic edges when $|\{f(v) : v \in A\}| \geqslant 2$ for all $A \in E$. For ordinary graphs, these notions coincide.

In the paper we investigate the asymptotic behavior of the strong chromatic number for the random k-uniform hypergraph in the classical binomial model $H(n, k, p)$. Recall that in this model every edge of $K_n^{(k)}$, the complete k-uniform hypergraph on n vertices, is included as an edge into $H(n, k, p)$ independently with probability $p \in (0, 1)$. In the case of graphs, when $k = 2$, $H(n, 2, p)$ is usually denoted by $G(n, p)$.

1.2 Related work: the chromatic number of random graphs

The chromatic number of the random graph $G(n, p)$ has been intensively studied for the last 40 years. Its asymptotic behavior was found by Bollobás [1] for the dense case (when $p = p(n)$ does not decrease fast enough) and by Łuczak [2] for all the remaining regimes up to the sparse one. The result of [2] says that if $p = p(n) = o(1)$ but $np \to +\infty$ with growth of n then the following convergence in probability holds:

$$\chi(G(n,p)) \cdot \frac{2\ln(np)}{np} \xrightarrow{\mathsf{Pr}} 1. \tag{1}$$

For the sparse case, when $np =$ is a positive constant, Łuczak [3] proved that the chromatic number of $G(n, p)$ has a limit two-valued distribution with two consecutive values. However, he didn't establish precisely these two values for given $c > 0$. This question was completely solved by Achlioptas and Naor [4] who showed that if r_c is the smallest integer r such that $c < 2r \ln r$ then

$$\mathsf{Pr}\left(\chi(G(n,p)) \in \{r_c, r_c + 1\}\right) \longrightarrow 1 \text{ as } n \to \infty. \tag{2}$$

It is easy to prove that for $c > 2r \ln r - \ln r$ the random graph $G(n, p)$ is not r-colorable with high probability. Together with (2) this establishes the one-point concentration of $\chi(G(n, p))$ roughly for the half of values of the parameter c:

- if $c \in (2(r-1)\ln(r-1), 2r \ln r - \ln r)$ then
$$\mathsf{Pr}\left(r \leqslant \chi(G(n, c/n)) \leqslant r + 1\right) \longrightarrow 1 \text{ as } n \to \infty;$$
- if $c \in (2r \ln r - \ln r, 2r \ln r)$ then
$$\mathsf{Pr}\left(\chi(G(n, c/n)) = r + 1\right) \longrightarrow 1 \text{ as } n \to \infty.$$

The above bounds for the r-colorability threshold were improved by Coja–Oghlan [5] (upper bound) and by Coja–Oghlan and Vilenchik [6] (lower bound). These results say that the chromatic number of $G(n, c/n)$

- does not exceed r with high probability if
$$c < 2\ln r - \ln r - 2\ln 2 - O\left(\frac{\ln r}{r}\right); \tag{3}$$
- is at least $r + 1$ with high probability if
$$c > 2\ln r - \ln r - 1 - o_r(1). \tag{4}$$

So, the r-colorability threshold as a threshold for the parameter c is established up to a bounded function. Moreover, for almost all values of c, the one-valued limit distribution of the chromatic number $\chi(G(n, c/n))$ is proved and the limit can be found explicitly as a function of c.

1.3 Related work: the strong chromatic number of random hypergraphs

The first results concerning the strong chromatic number of the random hypergraph $H(n,k,p)$ for fixed $k \geqslant 3$, were obtained in the 80-s. Schmidt [7] and Shamir [8] investigated the dense case when $p = p(n)$ does not decrease fast enough, Krivelevich and Sudakov [9] established the asymptotic behavior of $\chi_s(H(n,k,p))$ for all the remaining regimes up to the sparse one. The summary of the results can be represented as follows.

1. If $p = p(n) = \omega(\frac{\ln n}{n^{k-2}})$ then

$$\mathsf{Pr}\left(\chi_s(H(n,k,p)) = n\right) \longrightarrow 1 \text{ as } n \to \infty.$$

2. Let us denote $d = \binom{n-1}{k-1}p$, the expected value of a vertex degree in $H(n,k,p)$. If $d = d(n) \to +\infty$ and $d = o(n)$ (i.e. $p = \omega(n^{1-k})$ and $p = o(n^{2-k})$) then

$$\chi_s(H(n,k,p)) \cdot \frac{2\ln d}{(k-1)d} \xrightarrow{\mathsf{Pr}} 1 \text{ as } n \to \infty.$$

   This corresponds to (1) for random graphs.

3. If d is large enough, but fixed, then with probability tending to 1,

$$\frac{d(k-1)}{2\ln(d(k-1))} \leqslant \chi_s(H(n,k,p)) \leqslant \frac{d(k-1)}{2\ln(d(k-1))}\left(1 + \frac{1}{\ln^{0.1}(d(k-1))}\right). \qquad (5)$$

The last result is concerned with the sparse case, i.e. when $p = cn/\binom{n}{k}$ for given positive constant $c > 0$. This paper is the first attempt to improve it and to establish the bounds for the strong r-colorability threshold similar to (3) and (4).

Due to some technical reasons we deal only with the 3-uniform case in the current paper. We feel that the same approach can be applied to the general value of k, however some additional analysis should be involved.

1.4 New results

The main result of the paper provides the upper and lower bounds for the strong r-colorability threshold in the random 3-uniform hypergraph $H(n,3,p)$.

Theorem 1. *1) For given $r \geqslant 3$, if*

$$c > \frac{-\ln r}{\ln\left(1 - \frac{3}{r} + \frac{2}{r^2}\right)} = \frac{r\ln r}{3} - \frac{5}{18}\ln r + O\left(\frac{\ln r}{r}\right) \qquad (6)$$

then $\mathsf{Pr}\left(\chi_s(H(n,3,cn/\binom{n}{3})) > r\right) \to 1$ *as* $n \to \infty$.

2) There is an integer r_0 such that for any $r > r_0$ and any

$$c < \frac{r\ln r}{3} - \frac{5}{18}\ln r - \frac{1}{3} - r^{-1/6}, \qquad (7)$$

the following convergence holds: $\mathsf{Pr}\left(\chi_s(H(n,3,cn/\binom{n}{3})) \leqslant r\right) \to 1$ *as* $n \to \infty$.

The obtained bounds (6), (7) give the one-valued limit distribution for the strong chromatic number for almost all values of the parameter $c > 0$. The gap between (6) and (7) is a bounded function of r of the order $\frac{1}{3} + o_r(1)$.

Recently similar results were established for the weak chromatic number of the random hypergraph $H(n, k, p)$. Dyer, Frieze and Greenhill [10] proved the analogue of (2) for random hypergraphs and found the bounds for the r-colorability threshold with the gap of the order $O(\ln r)$. Ayre, Coja-Oghlan and Greenhill [11] improved the lower bound and reduced the gap with the upper estimate to a bounded value $\ln 2 + o_r(1)$ provided $r > r_0(k)$ (i.e. for r large enough in comparison with k). Shabanov [12] reduced the gap to a bounded value $\frac{r-1}{r} + o_{r,k}(1)$ for any values of r and k.

The remaining organization of the paper will be the following. In Section 2 we will prove the upper bound (6). Section 3 will be devoted to the proof of the lower bound (7).

2 Proof of the upper bound for the strong r-colorability threshold

The proof is based on the first moment argument. Consider another model of the random hypergraph: $H'(n, 3, m)$ in which m random independent edges of $K_n^{(3)}$ are chosen with replacement to be the edges of $H'(n, 3, m)$. Note that $H'(n, 3, m)$ does not necessarily have exactly m distinct edges. Let us choose $m = \lceil cn \rceil$ and let Z_n denote the number of edges of $H'(n, 3, \lceil cn \rceil)$. The following statement claims that it is sufficient to show that $H'(n, 3, \lceil cn \rceil)$ is not strongly r-colorable with high probability under the condition (6).

Claim 1. *For any $c' > c$,*

$$\mathsf{Pr}\left(\chi_s(H(n, 3, c'n/\binom{n}{3})) > r\right) \geqslant \mathsf{Pr}\left(\chi_s(H'(n, 3, \lceil cn \rceil) > r\right) + o(1).$$

Proof. Suppose ξ_n is a binomial random variable $Bin(\binom{n}{3}, c'n/\binom{n}{3})$ and let η_n be a random variable with the same distribution as Z_n. Then it is straightforward to check that

$$\mathsf{E}\xi_n = c'n,\ \mathsf{D}\xi_n = O(n),\ \mathsf{E}\eta_n \sim cn,\ \mathsf{D}\eta_n = O(n).$$

Hence, $\mathsf{Pr}(\xi_n > \eta_n) \to 1$ as $n \to \infty$.

Now let σ_n be a random ordering of the edges of $K_n^{(3)}$ independent both of ξ_n and η_n. Let us construct two random hypergraphs:

- H_1 takes the first ξ_n edges of $K_n^{(3)}$ according with σ_n;
- H_2 takes the first η_n edges of $K_n^{(3)}$ according with σ_n.

Clearly H_1 has the same distribution as $H(n, 3, c'n/\binom{n}{3})$ and H_2 has the same distribution as $H'(n, 3, \lceil cn \rceil)$. Moreover, $\mathsf{Pr}(H_2 \subset H_1) \to 1$ since $\xi_n > \eta_n$ with probability tending to 1. Thus,

$$\mathsf{Pr}\left(\chi_s(H(n, 3, c'n/\binom{n}{3})) > r\right) = \mathsf{Pr}(\chi_s(H_1) > r) \geqslant$$
$$\geqslant \mathsf{Pr}(\chi_s(H_2) > r) + o(1) = \mathsf{Pr}\left(\chi(H'(n, 3, \lceil cn \rceil) > r\right) + o(1).$$

□

Now we have to deal with $H'(n,3,\lceil cn\rceil)$. For given r-coloring τ of the vertex set, the probability that τ is a strong r-coloring for $H'(n,3,\lceil cn\rceil)$ is equal to

$$\mathsf{Pr}(\tau \text{ is a strong coloring for } H'(n,3,\lceil cn\rceil)) = \left(\sum_{\substack{i_1,i_2,i_3=1:\\ i_1<i_2<i_3}}^{r} \frac{v_{i_1}v_{i_2}v_{i_3}}{\binom{n}{3}} \right)^{\lceil cn\rceil},$$

where v_i is the cardinality of i-th color class of τ. The following statement claims that the above probability is maximized when all v_i are equal to n/r.

Claim 2. *For any coloring τ,*

$$\sum_{\substack{i_1,i_2,i_3=1:\\ i_1<i_2<i_3}}^{r} v_{i_1}v_{i_2}v_{i_3} \leqslant \frac{1}{6}r(r-1)(r-2)\left(\frac{n}{r}\right)^3.$$

Proof. Let us make the sum over the ordered triples:

$$\sum_{\substack{i_1,i_2,i_3=1:\\ i_1<i_2<i_3}}^{r} v_{i_1}v_{i_2}v_{i_3} = \frac{1}{6}\sum_{\substack{i_1,i_2,i_3=1:\\ i_1\neq i_2\neq i_3}}^{r} v_{i_1}v_{i_2}v_{i_3} = \left|\text{since } \sum_{i=1}^{r} v_i = n\right| =$$

$$= \frac{1}{6}\sum_{\substack{i_1,i_2=1:\\ i_1\neq i_2}}^{r} v_{i_1}v_{i_2}(n-v_{i_1}-v_{i_2}) = \frac{n}{6}\sum_{\substack{i_1,i_2=1:\\ i_1\neq i_2}}^{r} v_{i_1}v_{i_2} - \frac{2}{6}\sum_{\substack{i_1,i_2=1:\\ i_1\neq i_2}}^{r} v_{i_1}^2 v_{i_2} =$$

$$= \frac{n}{6}\sum_{i=1}^{r} v_i(n-v_i) - \frac{2}{6}\sum_{i=1}^{r} v_i^2(n-v_i) = \frac{1}{6}\left(n^3 - 3n\sum_{i=1}^{r} v_i^2 + 2\sum_{i=1}^{r} v_i^3\right).$$

The function $3nx^2 - 2x^3$ is nonnegative for $x \in [0,n]$ and is convex for $x \leqslant n/2$, so if every $v_i \leqslant n/2$ then by Jensen's inequality,

$$\left(n^3 - 3n\sum_{i=1}^{r} v_i^2 + 2\sum_{i=1}^{r} v_i^3\right) \leqslant n^3\left(1 - \frac{3}{r} + \frac{2}{r^2}\right) = r(r-1)(r-2)\left(\frac{n}{r}\right)^3.$$

If $v_i > n/2$ then $n^3 - 3nv_i^2 + 2v_i^3 \leqslant n^3/2$ which is smaller than the above expression for all $r \geqslant 6$. So, this case cannot provide the maximum value. For $r = 3,4,5$, one can use the method of Lagrange multipliers to show that the maximum corresponds to the situation of equal values $v_1 = \ldots = v_r = n/r$. □

Let us finish the proof. Let Y_n denote the number of the strong r-colorings of $H'(n,3,\lceil cn\rceil)$. Then

$$\mathsf{Pr}\left(\chi_s(H'(n,3,\lceil cn\rceil)) \leqslant r\right) \leqslant \mathsf{E}Y_n \leqslant r^n \left(\frac{r(r-1)(r-2)\left(\frac{n}{r}\right)^3}{6\binom{n}{3}}\right)^{\lceil cn\rceil} =$$

$$= r^n\left(1 - \frac{3}{r} + \frac{2}{r^2} + O\left(\frac{1}{n}\right)\right)^{\lceil cn\rceil} =$$

$$= \exp\left[n\left(\ln r + c\ln\left(1 - \frac{3}{r} + \frac{2}{r^2}\right)\right) + O(1)\right].$$

The condition (6) implies that the above estimate tends to 0 with growth of n and, consequently, the same does the probability of the strong r-colorability of $H'(n,3,\lceil cn \rceil)$. It remains to note that for large r,

$$\frac{-\ln r}{\ln\left(1-\frac{3}{r}+\frac{2}{r^2}\right)} = \frac{-\ln r}{-\frac{3}{r}+\frac{2}{r^2}-\frac{9}{2r^2}+O\left(\frac{1}{r^3}\right)} =$$
$$= \frac{r\ln r}{3}\left(1+\frac{5}{6r}+O\left(\frac{1}{r^2}\right)\right)^{-1} = \frac{r\ln r}{3}\left(1-\frac{5}{6r}+O\left(\frac{1}{r^2}\right)\right) =$$
$$= \frac{r\ln r}{3} - \frac{5}{18}\ln r + O\left(\frac{\ln r}{r}\right).$$

3 Proof of the lower bound for the strong r-colorability threshold

The proof is based on the application of the second moment method. We follow the general scheme from [10] and the analytic approach from [12]. However, we had to derive new ideas to obtain the bound (7) since the result [12] does not work for 3-uniform hypergraphs.

3.1 First step: the sharp threshold

The first thing we have to do in the second moment method is to establish that the property of the strong r-colorability has a sharp threshold in $H(n,3,p)$ model. The latter means that there is a function $\widehat{p}=\widehat{p}(n)$ such that for any fixed $\varepsilon>0$,

$$\mathsf{Pr}\left(\chi_s(H(n,3,p))\leqslant r\right) = \begin{cases} 1, & \text{if } p\leqslant(1-\varepsilon)\widehat{p}; \\ 0, & \text{if } p\geqslant(1+\varepsilon)\widehat{p}. \end{cases}$$

The existence of the sharp threshold follows form the general result of Hatami and Molloy [13] concerning the hypergraph properties. If a k-uniform hypergraph H has the property A if and only if there is a homomorphism from H to a fixed connected k-uniform hypergraph F, $k\geqslant 3$, then the property A has the sharp threshold in the binomial model $H(n,k,p)$ of the random hypergraph. It is easy to see that a 3-uniform hypergraph H admits a strong r-coloring if and only if there is a homomorphism from H to $F=(V,E)$ where

$$V=\{1,\ldots,r\},\quad E=\{(v,u,w): 1\leqslant v<u<w\leqslant r\}.$$

The existence of the sharp threshold can be applied as follows. If for $c=\frac{r\ln r}{3}-\frac{5}{18}\ln r-\frac{1}{3}-r^{-1/6}$, we show that

$$\liminf_{n\to\infty}\mathsf{Pr}\left(\chi_s(H(n,3,cn/\binom{n}{3}))\right)>0 \tag{8}$$

then the value $p=cn/\binom{n}{k}$ cannot be greater than the threshold probability. Hence for any $c'<c$, the probability $\mathsf{Pr}\left(\chi_s(H(n,3,c'n/\binom{n}{k}))\right)$ tends to 1 with growth of n. So it is sufficient to prove (8) and the lower bound (7) immediately follows.

3.2 Second step: another model

As in the proof of the upper bound it will be more convenient to use another model of the random hypergraph. Let $H(n,3,m)$ denote the random hypergraph consisting of m independent edges, in which every edge contains 3 random vertices chosen with replacement at random. Since all the vertices are chosen independently, $H(n,3,m)$ can contain non-proper edges with two or three coinciding vertices and it also can contain duplicate edges with the same set of vertices. Recall that under the strong coloring

- an edge of size one is always colored properly,
- an edge of size two has two vertices colored with distinct colors.

Let $m = \lceil cn \rceil$ with $c = \frac{r \ln r}{3} - \frac{5}{18} \ln r - \frac{1}{3} - r^{-1/6}$ and let Y_n denote the number of distinct proper edges of $H(n,3,m)$. It is easy to see that

$$\mathsf{E} Y_n = cn + o(n) \text{ and } \mathsf{D} Y_n = O(n).$$

Thus by using the same argument as in the proof of Claim 1 one can show that for any $c' < c$,

$$\mathsf{Pr}\left(\chi_s(H(n,3,c'n/\binom{n}{3})) \leqslant r\right) \geqslant \mathsf{Pr}\left(\chi_s(H(n,3,m)) \leqslant r\right) + o(1). \tag{9}$$

If we show that

$$\liminf_{n\to\infty} \mathsf{Pr}\left(\chi_s(H(n,3,m)) \leqslant r\right) > 0 \tag{10}$$

then (9) will immediately imply (8). Finally, it was shown in [10] (see Lemma 1.4 in [10]) that it is sufficient to prove the above inequality only for the subsequence of the type $n = rn'$, i.e. we may consider only the case when n is divisible by r.

3.3 Third step: balanced colorings

Now let n be divisible by r and let X_n denote the number of strong balanced r-colorings for the random hypergraph $H(n,3,m)$. Recall that an r-coloring is said to be *balanced* if all the color classes of this coloring have the same cardinality n/r. Clearly,

$$\mathsf{Pr}\left(\chi_s(H(n,3,m)) \leqslant r\right) \geqslant \mathsf{Pr}\left(X_n > 0\right).$$

The Paley–Zygmund inequality implies that

$$\mathsf{Pr}\left(X_n > 0\right) \geqslant \frac{(\mathsf{E} X_n)^2}{\mathsf{E} X_n^2},$$

so to establish (10) it is sufficient to show that there exists a function $C(r) > 0$ such that for all large enough n,

$$\mathsf{E} X_n^2 \leqslant C(r) \cdot (\mathsf{E} X_n)^2, \tag{11}$$

i.e. the ratio of the second moment to the square of the first moment is bounded from above by some function depending only on r.

3.4 Fourth step: calculation of moments

Now we have to calculate the first two moments of X_n. Everywhere below we assume that n is large enough in comparison with r. Let τ be an arbitrary balanced r-coloring of $H(n,3,m)$. The independence of the edges of $H(n,3,m)$ implies the following formula for the expected value of X_n:

$$\begin{aligned}\mathsf{E}X_n &= \frac{n!}{((n/r)!)^r}\left(\frac{r(r-1)(r-2)}{r^3}+3\frac{r(r-1)}{r^2n}+\frac{1}{n^2}\right)^m = \\ &= |\text{since } m=\lceil cn\rceil| = \Theta\left(n^{1/2-r/2}r^n\left(\frac{r(r-1)(r-2)}{r^3}+O\left(\frac{1}{n}\right)\right)^{cn}\right) = \\ &= \Theta\left(n^{1/2-r/2}\exp\left[n\left(\ln r + c\ln\left(\frac{(r-1)(r-2)}{r^2}\right)\right)\right]\right). \qquad (12)\end{aligned}$$

Now let us calculate the second moment of X_n. Suppose τ and τ' are two balanced r-colorings of the vertex set. Let $A=A(\tau,\tau')$ denote the *colorings matrix*, i.e. $A=\|a_{ij}\|_{i,j=1}^r$, where a_{ij} is the number of vertices colored with color i in the coloring τ and with color j in the coloring τ'. Since both colorings are balanced we have the following properties for A: for every $i,j=1,\dots,r$,

$$\sum_{j'=1}^r a_{ij'} = \frac{n}{r}, \quad \sum_{i'=1}^r a_{i'j} = \frac{n}{r}, \quad a_{ij}\geqslant 0. \qquad (13)$$

Let $\mathcal{A}$ denote the set of such matrices. Hence the probability that both τ and τ' are the strong colorings for $H(n,3,m)$ is equal to

$$\left(\sum_{\substack{i_1\neq i_2\neq i_3;\\ j_1\neq j_2\neq j_3}}\frac{a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}}{n^3}+3\sum_{\substack{i_1\neq i_2;\\ j_1\neq j_2}}\frac{a_{i_1j_1}a_{i_2j_2}}{n^3}+\frac{1}{n^2}\right)^m.$$

Thus

$$\begin{aligned}\mathsf{E}X_n^2 &= \sum_{A\in\mathcal{A}}\frac{n!}{\prod_{i,j=1}^r a_{ij}!}\left(\sum_{\substack{i_1\neq i_2\neq i_3;\\ j_1\neq j_2\neq j_3}}\frac{a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}}{n^3}+3\sum_{\substack{i_1\neq i_2;\\ j_1\neq j_2}}\frac{a_{i_1j_1}a_{i_2j_2}}{n^3}+\frac{1}{n^2}\right)^m = \\ &= \Theta\left(\sum_{A\in\mathcal{A}}\frac{\sqrt{n}}{\prod_{i,j=1}^r\sqrt{a_{ij}+1}}e^{-\sum_{i,j=1}^r a_{ij}\ln a_{ij}}\left(\sum_{\substack{i_1\neq i_2\neq i_3;\\ j_1\neq j_2\neq j_3}}\frac{a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}}{n^3}\right)^{cn}\right) \\ &= |\text{using } (12)| = \Theta\left((\mathsf{E}X_n)^2 n^{r-1/2}\sum_{A\in\mathcal{A}}\frac{1}{\prod_{i,j=1}^r\sqrt{a_{ij}+1}}\exp\left[n\left(g(A)-f(A)\right)\right]\right), \qquad (14)\end{aligned}$$

where

$$f(A)=\sum_{i,j=1}^r\frac{a_{ij}}{n}\ln(r^2a_{ij}), \quad g(A)=c\ln\left(\left(\frac{r^2}{(r-1)(r-2)}\right)^2\sum_{\substack{i_1\neq i_2\neq i_3;\\ j_1\neq j_2\neq j_3}}\frac{a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}}{n^3}\right). \qquad (15)$$

The following lemma is crucial.

Lemma 1. *There is a positive function $b = b(r) > 0$ and an integer r_0 such that for any $r > r_0$ and $c = \frac{r \ln r}{3} - \frac{5}{18} \ln r - \frac{1}{3} - r^{-1/6}$, the following property holds: for every matrix $A \in \mathcal{A}$,*

$$f(A) - g(A) \geqslant b \sum_{i,j=1}^{r} \left(\frac{a_{ij}}{n} - \frac{1}{r^2} \right)^2. \tag{16}$$

The proof of Lemma 1 will be given in the next section. Now let us finish the proof of the main theorem.

The relation (14) implies that for some absolute constant $D > 0$,

$$\frac{\mathsf{E} X_n^2}{(\mathsf{E} X_n)^2} \leqslant D \cdot n^{r-1/2} \sum_{A \in \mathcal{A}} \frac{1}{\prod_{i,j=1}^{r} \sqrt{a_{ij}+1}} \exp\left[n\left(g(A) - f(A)\right)\right] \leqslant$$
$$\leqslant D \cdot n^{r-1/2} \sum_{A \in \mathcal{A}} \frac{1}{\prod_{i,j=1}^{r} \sqrt{a_{ij}+1}} \exp\left[-nb \sum_{i,j=1}^{r} \left(\frac{a_{ij}}{n} - \frac{1}{r^2} \right)^2\right].$$

Note that due to (13) the sum over $\mathcal{A}$ does not exceed the sum over a_{ij} (from 0 to n/r) for $i,j \leqslant r-1$. Furthermore, $(\sqrt{a+1})^{-1} \exp\left[-nb\left(\frac{a}{n} - \frac{1}{r^2}\right)^2\right] = O_r(n^{-1/2})$ for any $a \in [0, n/r]$. Application of this bound to a_{ij} for $\max(i,j) = r$ yields the following estimate:

$$\frac{\mathsf{E} X_n^2}{(\mathsf{E} X_n)^2} = O_r\left(n^{r-1/2} n^{-1/2(r^2-(r-1)^2)} \sum_{A \in \mathcal{A}} \frac{1}{\prod_{i,j=1}^{r-1} \sqrt{a_{ij}+1}} \exp\left[-nb \sum_{i,j=1}^{r-1} \left(\frac{a_{ij}}{n} - \frac{1}{r^2} \right)^2\right] \right) \leqslant$$
$$\leqslant O_r\left(\sum_{i,j=1}^{r-1} \sum_{a_{ij}=0}^{n/r} \frac{1}{\prod_{i,j=1}^{r-1} \sqrt{a_{ij}+1}} \exp\left[-nb \sum_{i,j=1}^{r-1} \left(\frac{a_{ij}}{n} - \frac{1}{r^2} \right)^2\right] \right) =$$
$$= O_r\left(\prod_{i,j=1}^{r-1} \left(\sum_{a_{ij}=0}^{n/r} \frac{1}{\sqrt{a_{ij}+1}} \exp\left[-nb\left(\frac{a_{ij}}{n} - \frac{1}{r^2} \right)^2\right] \right) \right) =$$
$$= O_r\left(\left(\sum_{a=0}^{n/r} \frac{1}{\sqrt{a+1}} \exp\left[-nb\left(\frac{a}{n} - \frac{1}{r^2} \right)^2\right] \right)^{(r-1)^2} \right) =$$
$$= O_r\left(\left(\int_0^{+\infty} \frac{1}{\sqrt{x+1}} \exp\left[-nb\left(\frac{x}{n} - \frac{1}{r^2} \right)^2\right] dx \right)^{(r-1)^2} \right) = C(r).$$

The required relation (11) is established. Theorem 1 is proved.

3.5 Final step: proof of Lemma 1

It will be convenient to use the notation $\varepsilon_{ij} = \frac{a_{ij}}{n} - \frac{1}{r^2}$, $i,j = 1, \ldots, r$. Then (13) implies the following conditions on $\varepsilon = (\varepsilon_{ij} : \ i,j = 1, \ldots, r)$: for every $i,j = 1, \ldots, r$,

$$\sum_{j'=1}^{r} \varepsilon_{ij'} = 0, \quad \sum_{i'=1}^{r} \varepsilon_{i'j} = 0, \quad \varepsilon_{ij} \in \left[-\frac{1}{r^2}, \frac{1}{r} - \frac{1}{r^2} \right]. \tag{17}$$

Hence the functions $f(A)$ and $g(A)$ from (15) can be represented as functions of ε:

$$f(A) = \sum_{i,j=1}^{r} \frac{a_{ij}}{n} \ln(r^2 a_{ij}) = \sum_{i,j=1}^{r} \left(\frac{1}{r^2} + \varepsilon_{ij}\right) \ln(1 + r^2\varepsilon_{ij}),$$

$$g(A) = c\ln\left(\left(\frac{r^2}{(r-1)(r-2)}\right)^2 \sum_{\substack{i_1\neq i_2\neq i_3;\\ j_1\neq j_2\neq j_3}} \left(\frac{1}{r^2} + \varepsilon_{i_1j_1}\right)\left(\frac{1}{r^2} + \varepsilon_{i_2j_2}\right)\left(\frac{1}{r^2} + \varepsilon_{i_3j_3}\right)\right).$$

Let us consider the last expression in more details.

Claim 3.

$$g(A) = c\ln\left(1 + 3\left(\frac{r}{r-1}\right)^2 \sum_{i,j=1}^{r} \varepsilon_{ij}^2 + 4\left(\frac{r^2}{(r-1)(r-2)}\right)^2 \sum_{i,j=1}^{r} \varepsilon_{ij}^3\right).$$

Proof. Clearly,

$$\sum_{\substack{i_1\neq i_2\neq i_3;\\ j_1\neq j_2\neq j_3}} \left(\frac{1}{r^2} + \varepsilon_{i_1j_1}\right)\left(\frac{1}{r^2} + \varepsilon_{i_2j_2}\right)\left(\frac{1}{r^2} + \varepsilon_{i_3j_3}\right) = \frac{r^2(r-1)^2(r-2)^2}{r^6} +$$

$$+3\frac{(r-1)^2(r-2)^2}{r^4} \sum_{i_1,j_1=1}^{r} \varepsilon_{i_1j_1} + 3\frac{(r-2)^2}{r^2} \sum_{i_1\neq i_2; j_1\neq j_2} \varepsilon_{i_1j_1}\varepsilon_{i_2j_2} + \sum_{\substack{i_1\neq i_2\neq i_3;\\ j_1\neq j_2\neq j_3}} \varepsilon_{i_1j_1}\varepsilon_{i_2j_2}\varepsilon_{i_3j_3}.$$

Using the conditions (17) we obtain the following equalities:

$$\sum_{i_1,j_1=1}^{r} \varepsilon_{i_1j_1} = 0; \quad \sum_{i_1\neq i_2; j_1\neq j_2} \varepsilon_{i_1j_1}\varepsilon_{i_2j_2} = \sum_{i,j=1}^{r} \varepsilon_{ij}^2; \quad \sum_{\substack{i_1\neq i_2\neq i_3;\\ j_1\neq j_2\neq j_3}} \varepsilon_{i_1j_1}\varepsilon_{i_2j_2}\varepsilon_{i_3j_3} = 4\sum_{i,j=1}^{r} \varepsilon_{ij}^3.$$

Their substitution implies the required representation for $g(A)$. □

Now, let us introduce the partial sums: for $i = 1, \ldots, r$, we denote

$$f_i(A) = \sum_{j=1}^{r} \left(\frac{1}{r^2} + \varepsilon_{ij}\right) \ln(1 + r^2\varepsilon_{ij}),$$

$$g_i(A) = c\left(3\left(\frac{r}{r-1}\right)^2 \sum_{j=1}^{r} \varepsilon_{ij}^2 + 4\left(\frac{r^2}{(r-1)(r-2)}\right)^2 \sum_{j=1}^{r} \varepsilon_{ij}^3\right). \tag{18}$$

It is easy to see that $f(A) = \sum_{i=1}^{r} f_i(A)$ and $g(A) \leqslant \sum_{i=1}^{r} g_i(A)$. Our strategy is to estimate the difference $f_i(A) - g_i(A)$ for different cases. Clearly, both $f_i(A)$ and $g_i(A)$ depend only on the vector $\varepsilon_i = (\varepsilon_{ij}, j = 1, \ldots, r)$. We say that ε_i

- is a *good* vector if $\varepsilon_{ij} \leqslant \frac{1}{r} - \frac{1}{r^2} - \frac{2}{r\ln r}$ for every $j = 1, \ldots, r$;
- is a *normal* vector if there is j_0 such that $\varepsilon_{ij_0} \in [\frac{1}{r} - \frac{1}{r^2} - \frac{2}{r\ln r}, \frac{1}{r} - \frac{1}{r^2} - r^{-7/4}]$;
- is a *bad* vector if there is j_0 such that $\varepsilon_{ij_0} > \frac{1}{r} - \frac{1}{r^2} - r^{-7/4}$.

Now we consider these cases separately.

3.5.1 Good vectors

For a good vector ε_i, we prove the following claim.

Claim 4. *If r is large enough and ε_i is a good vector then*

$$f_i(A) - g_i(A) \geqslant \frac{r^2}{4} \sum_{j:\varepsilon_{ij}<0} \varepsilon_{ij}^2 + \frac{r}{2} \sum_{j:\varepsilon_{ij}>0} \varepsilon_{ij}^2. \tag{19}$$

Proof. Since every ε_{ij} does not exceed $\frac{1}{r} - \frac{1}{r^2} - \frac{2}{r\ln r}$ and $c < \frac{r\ln r}{3}$, the representation (18) implies that

$$g_i(A) \leqslant c\left(3\left(1 + \frac{2}{r} + O(r^{-2})\right)\sum_{j=1}^{r} \varepsilon_{ij}^2 + \left(\frac{4}{r} + O(r^{-2})\right)\sum_{j=1}^{r} \varepsilon_{ij}^2\right) =$$
$$= c\left(3 + \frac{10}{r} + O(r^{-2})\right)\sum_{j=1}^{r} \varepsilon_{ij}^2 \leqslant \left(r\ln r + \frac{10}{3}\ln r + O\left(\frac{\ln r}{r}\right)\right)\sum_{j=1}^{r} \varepsilon_{ij}^2.$$

Now let us estimate $f_i(A)$. For negative $x \geqslant -1$, we can apply the inequality $(1+x)\ln(1+x) \geqslant x + x^2/2$. Thus, for $\varepsilon_{ij} < 0$, we have

$$\left(\frac{1}{r^2} + \varepsilon_{ij}\right)\ln(1 + r^2\varepsilon_{ij}) \geqslant \varepsilon_{ij} + \frac{1}{r^2}\frac{(r^2\varepsilon_{ij})^2}{2} = \varepsilon_{ij} + \frac{r^2}{2}\varepsilon_{ij}^2.$$

If $\varepsilon_{ij} > 0$ but $\varepsilon_{ij} \leqslant \frac{1}{r\ln r} - \frac{1}{r^2}$ then we can use the inequality $(1+x)\ln(1+x) > x + \frac{x^2}{2(1+x/3)}$ for $x > 0$. Hence

$$(1/r^2 + \varepsilon_{ij})\ln(1 + r^2\varepsilon_{ij}) \geqslant \varepsilon_{ij} + \frac{r^2\varepsilon_{ij}^2}{2(1 + r^2\varepsilon_{ij}/3)} \geqslant$$
$$\geqslant |(\text{since } 1 + r^2\varepsilon_{ij}/3 < 1 + \tfrac{r}{3\ln r})| \geqslant$$
$$\geqslant \varepsilon_{ij} + \frac{3r\ln r}{2}\left(1 + O\left(\frac{\ln r}{r}\right)\right)\varepsilon_{ij}^2.$$

If $\varepsilon_{ij} \in (\frac{1}{r\ln r} - \frac{1}{r^2}, \frac{1}{r} - \frac{1}{r^2} - \frac{2\ln\ln r}{r\ln r}))$ then we estimate the summand as follows:

$$(1/r^2 + \varepsilon_{ij})\ln(1 + r^2\varepsilon_{ij}) \geqslant (1/r^2 + \varepsilon_{ij})\ln\left(\frac{r}{\ln r}\right) \geqslant \varepsilon_{ij} + \varepsilon_{ij}(\ln r - \ln\ln r - 1) \geqslant$$
$$\geqslant |\text{since } 1 \geqslant r\varepsilon_{ij}\left(1 - \tfrac{1}{r} - \tfrac{2\ln\ln r}{\ln r}\right)^{-1}| \geqslant$$
$$\geqslant \varepsilon_{ij} + r\varepsilon_{ij}^2\left(1 - \frac{1}{r} - \frac{2\ln\ln r}{\ln r}\right)^{-1}(\ln r - \ln\ln r - 1) =$$
$$= \varepsilon_{ij} + r\ln r\cdot\varepsilon_{ij}^2\left(1 + \frac{2\ln\ln r}{\ln r} + O\left(\left(\frac{\ln\ln r}{\ln r}\right)^2\right)\right)\left(1 - \frac{\ln\ln r}{\ln r} - \frac{1}{\ln r}\right) =$$
$$= \varepsilon_{ij} + r\ln r\cdot\varepsilon_{ij}^2\left(1 + \frac{\ln\ln r}{\ln r} + O\left(\left(\frac{\ln\ln r}{\ln r}\right)^2\right)\right).$$

Finally, if $\varepsilon_{ij} \in (\frac{1}{r} - \frac{1}{r^2} - \frac{2\ln\ln r}{r\ln r}, \frac{1}{r} - \frac{1}{r^2} - \frac{2}{r\ln r})$ then

$$
\begin{aligned}
&(1/r^2 + \varepsilon_{ij}) \ln(1 + r^2\varepsilon_{ij}) \geqslant (1/r^2 + \varepsilon_{ij}) \left(\ln r + \ln \left(1 - \frac{2\ln\ln r}{\ln r} \right) \right) \geqslant \\
&\geqslant \varepsilon_{ij} + \varepsilon_{ij}(\ln r - o(1) - 1) \geqslant \\
&\geqslant |\text{since } 1 \geqslant r\varepsilon_{ij} \left(1 - \tfrac{1}{r} - \tfrac{2}{r\ln r}\right)^{-1}| \geqslant
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \varepsilon_{ij} + r\varepsilon_{ij}^2 \left(1 - \frac{1}{r} - \frac{2}{\ln r} \right)^{-1} (\ln r - o(1) - 1) = \\
&= \varepsilon_{ij} + r\ln r \cdot \varepsilon_{ij}^2 \left(1 + \frac{2}{\ln r} + O\left(\frac{1}{\ln r}\right)^2 \right) \left(1 - o((\ln r)^{-1}) - \frac{1}{\ln r} \right) = \\
&= \varepsilon_{ij} + r\ln r \cdot \varepsilon_{ij}^2 \left(1 + \frac{1}{\ln r} + o\left(\frac{1}{\ln r}\right) \right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f_i(A) &\geqslant \sum_{j:\varepsilon_{ij}<0} \left(\varepsilon_{ij} + \frac{r^2}{2}\varepsilon_{ij}^2 \right) + \sum_{j:\varepsilon_{ij}>0} \left(\varepsilon_{ij} + r\ln r \cdot \varepsilon_{ij}^2 \left(1 + \frac{1}{\ln r} + o\left(\frac{1}{\ln r}\right) \right) \right) = \\
&= \frac{r^2}{2} \sum_{j:\varepsilon_{ij}<0} \varepsilon_{ij}^2 + (r\ln r + r + o(r)) \sum_{j:\varepsilon_{ij}>0} \varepsilon_{ij}^2.
\end{aligned}
$$

The above bound for $f_i(A)$ together with the estimate for $g_i(A)$ implies the required relation (19): for large enough r,

$$
\begin{aligned}
f_i(A) - g_i(A) &\geqslant \left(\frac{r^2}{2} - O(r\ln r) \right) \sum_{j:\varepsilon_{ij}<0} \varepsilon_{ij}^2 + (r + o(r) - O(\ln r)) \sum_{j:\varepsilon_{ij}>0} \varepsilon_{ij}^2 \geqslant \\
&\geqslant \frac{r^2}{4} \sum_{j:\varepsilon_{ij}<0} \varepsilon_{ij}^2 + \frac{r}{2} \sum_{j:\varepsilon_{ij}>0} \varepsilon_{ij}^2.
\end{aligned}
$$

□

3.5.2 Normal vectors

Let ε_i be a normal vector. Recall that there is j_0 such that $\varepsilon_{ij_0} \in [\frac{1}{r} - \frac{1}{r^2} - \frac{2}{r\ln r}, \frac{1}{r} - \frac{1}{r^2} - r^{-7/4}]$. Note that such j_0 can be only one and all the others ε_{ij} should be less than $\frac{1}{r} - \frac{1}{r^2} - \frac{2}{r\ln r}$. Indeed, if there is $j' \neq j_0$ with $\varepsilon_{ij'} \geqslant \frac{1}{r} - \frac{1}{r^2} - \frac{2}{r\ln r}$ then

$$
\sum_{j=1}^{r} \varepsilon_{ij} \geqslant \frac{2}{r} - \frac{2}{r^2} - \frac{4}{r\ln r} - \frac{r-2}{r^2} = \frac{1}{r} - \frac{4}{r\ln r} > 0.
$$

A contradiction with (17). Without loss of generation, we may assume that $j_0 = i$.

For a normal vector, we have the following estimate of $f_i(A) - g_i(A)$.

Claim 5. *If r is large enough and ε_i is a normal vector then*

$$f_i(A) - g_i(A) \geqslant \frac{1}{8} r^{-7/4} \ln r. \tag{20}$$

Proof. Let us use the notation $\delta_i = \frac{1}{r} - \frac{1}{r^2} - \varepsilon_{ii}$. Thus, $\delta_i \in [r^{-7/4}, \frac{2}{r \ln r}]$. Let us also denote $\delta_{ij} = 1/r^2 + \varepsilon_{ij}$ for $j \neq i$. Then all δ_{ij} are nonnegative and δ_i is their sum.

Let us estimate $f_i(A)$:

$$\begin{aligned}
f_i(A) = \sum_{j=1}^{r} (1/r^2 + \varepsilon_{ij}) \ln(1 + r^2 \varepsilon_{ij}) = (1/r - \delta_i) \ln(r - r^2 \delta_i) + \sum_{j \neq i}^{r} \delta_{ij} \ln(r^2 \delta_{ij}) = \\
= \frac{\ln r}{r} - \ln r \delta_i + (1/r - \delta_i) \ln(1 - r\delta_i) + 2 \ln r \sum_{j \neq i}^{r} \delta_{ij} + + \sum_{j \neq i}^{r} \delta_{ij} \ln \delta_{ij} \geqslant \\
\geqslant |\text{by Jensen's inequality the last sum is at least } - \ln(r-1)\delta_i + \delta_i \ln \delta_i| \geqslant \\
\geqslant \frac{\ln r}{r} + \ln r \delta_i + (1/r - \delta_i) \ln(1 - r\delta_i) - \ln(r-1)\delta_i + \delta_i \ln \delta_i \geqslant \\
\geqslant |\text{since } \ln(1 - r\delta_i) > (-r\delta_i)/(1 - r\delta_i)| \geqslant \\
\geqslant \frac{\ln r}{r} + \delta_i \ln \delta_i + \ln \left(\frac{r}{r-1} \right) \delta_i - \delta_i.
\end{aligned} \tag{21}$$

Now we proceed to $g_i(A)$. We have to estimate $\sum_{j=1}^{r} \varepsilon_{ij}^2$ and $\sum_{j=1}^{r} \varepsilon_{ij}^3$:

$$\begin{aligned}
\sum_{j=1}^{r} \varepsilon_{ij}^2 = \left(\frac{1}{r} - \frac{1}{r^2} - \delta_i \right)^2 + \sum_{j \neq i} \left(\frac{1}{r^2} - \delta_{ij} \right)^2 = \\
= \frac{1}{r^2} + \frac{1}{r^4} - \frac{2}{r^3} - \frac{2\delta_i}{r} + \frac{2\delta_i}{r^2} + \delta_i^2 + \frac{r-1}{r^4} - \frac{2\delta_i}{r^2} + \sum_{j \neq i} \delta_{ij}^2 \leqslant \\
\leqslant |\text{since } \textstyle\sum_{j \neq i} \delta_{ij}^2 \leqslant \delta_i^2| \leqslant \\
\leqslant \frac{1}{r^2} - \frac{1}{r^3} - \frac{2\delta_i}{r} + 2\delta_i^2.
\end{aligned}$$

The dominating summand is r^{-2}, hence the sum of cubes is $O(r^{-3})$. Furthermore,

$$\begin{aligned}
g_i(A) \leqslant 3c \left(1 + O\left(\frac{1}{r} \right) \right) \sum_{j=1}^{r} \varepsilon_{ij}^2 = 3c \left(\frac{1}{r^2} - \frac{2\delta_i}{r} + 2\delta_i^2 + O\left(\frac{1}{r^3} \right) \right) \leqslant \\
\leqslant |\text{since } c < \tfrac{r \ln r}{3}| \leqslant \frac{\ln r}{r} - 2 \ln r \cdot \delta_i + 2\delta_i^2 (r \ln r) + O\left(\frac{\ln r}{r^2} \right).
\end{aligned}$$

Note that $r\delta_i = o_r(1)$ and $\delta_i \gg r^{-2}$, so $g_i(A) \leqslant \frac{\ln r}{r} - 2 \ln r \cdot \delta_i (1 + o(1))$. This estimate together with the lower bound (21) implies the required inequality:

$$\begin{aligned}
f_i(A) - g_i(A) \geqslant \delta_i \ln \delta_i - \delta_i + \ln \left(\frac{r}{r-1} \right) \delta_i + 2 \ln r \cdot \delta_i (1 + o(1)) \geqslant \\
\geqslant |\text{since } \ln \delta_i \geqslant -\tfrac{7}{4} \ln r| \geqslant \\
\geqslant \frac{1}{4} \cdot \ln r \cdot \delta_i (1 + o(1)) \geqslant \frac{1}{8} r^{-7/4} \ln r.
\end{aligned}$$

□

3.5.3 Bad vectors

It remains to consider bad vectors. Recall that in every bad vector ε_i there is an index j_0 such that $\varepsilon_{ij_0} > \frac{1}{r} - \frac{1}{r^2} - r^{-7/4}$. Again, such an index can be only one in any bad vector. Moreover, if $\varepsilon_{i'}$ is another bad vector then the condition (17) implies that the largest elements of ε_i and $\varepsilon_{i'}$ cannot lie in the same column. So, without loss of generality, we may assume that the largest elements are diagonal, i.e. $j_0 = i$ for every bad vector ε_i.

Now we will consider all the bad vectors simultaneously. Let $I \subset \{1, \dots, r\}$ denote the set of indexes of the bad vectors. Let us introduce the following values:

$$f'(A) = \sum_{i \in I} f_i(A),$$

$$g'(A) = c \ln \left(1 + \sum_{i \in I} \sum_{j=1}^{r} \left[3 \left(\frac{r}{r-1} \right)^2 \varepsilon_{ij}^2 + 4 \left(\frac{r^2}{(r-1)(r-2)} \right)^2 \varepsilon_{ij}^3 \right] \right).$$

Their difference can be estimated as follows.

Claim 6. *If r is large enough then*

$$f'(A) - g'(A) \geqslant \frac{3|I| \ln r}{2r^2} \left(\frac{|I|}{r} - 1 \right) + \frac{3|I|}{r^{2+1/3}} + O\left(\frac{\ln r}{r^{3/2}} \right). \tag{22}$$

Proof. Once again we denote $\delta_i = \frac{1}{r} - \frac{1}{r^2} - \varepsilon_{ii}$ and $\delta_{ij} = \frac{1}{r^2} + \varepsilon_{ij}$ for $j \neq i$. We will consider the sum over I under the logarithm in the expression for $g'(A)$ and apply (21) to estimate $f'(A)$ (note that (21) holds for bad vectors also).

Let us make the Taylor expansion to understand the order of sums $\sum_{j=1}^{r} \varepsilon_{ij}^2$ and $\sum_{j=1}^{r} \varepsilon_{ij}^3$:

$$\begin{aligned}
\sum_{j=1}^{r} \varepsilon_{ij}^2 &= \left(\frac{1}{r} - \frac{1}{r^2} - \delta_i \right)^2 + \sum_{j \neq i} \left(\frac{1}{r^2} - \delta_{ij} \right)^2 = \\
&= \frac{1}{r^2} + \frac{1}{r^4} - \frac{2}{r^3} - \frac{2\delta_i}{r} + \frac{2\delta_i}{r^2} + \delta_i^2 + \frac{r-1}{r^4} - \frac{2\delta_i}{r^2} + \sum_{j \neq i} \delta_{ij}^2 = \\
&= \text{(using the fact that } \textstyle\sum_{j \neq i} \delta_{ij}^2 \leq \delta_i^2 = O(r^{-7/2})) \\
&= \frac{1}{r^2} - \frac{1}{r^3} - \frac{2\delta_i}{r} + O(r^{-7/2}).
\end{aligned}$$

The sum of cubes can be represented in a similar way:

$$\sum_{j=1}^{r} \varepsilon_{ij}^3 = \left(\frac{1}{r} - \frac{1}{r^2} - \delta_i \right)^3 + \sum_{j \neq i} \left(\frac{1}{r^2} - \delta_{ij} \right)^3 = \frac{1}{r^3} + O(r^{-15/4}).$$

Furthermore,

$$\sum_{j=1}^{r}\left[3\left(\frac{r}{r-1}\right)^2\varepsilon_{ij}^2+4\left(\frac{r^2}{(r-1)(r-2)}\right)^2\varepsilon_{ij}^3\right]=$$

$$=3\left(1+\frac{2}{r}+O\left(\frac{1}{r^2}\right)\right)\left(\frac{1}{r^2}-\frac{1}{r^3}-\frac{2\delta_i}{r}+O(r^{-7/2})\right)+$$

$$+4\left(1+O\left(\frac{1}{r}\right)\right)\left(\frac{1}{r^3}+O(r^{-15/4})\right)=$$

$$=3\left(\frac{1}{r^2}-\frac{1}{r^3}-\frac{2\delta_i}{r}+O(r^{-7/2})+\frac{2}{r^3}\right)+\frac{4}{r^3}+O(r^{-15/4})=$$

$$=\frac{3}{r^2}+\frac{7}{r^3}-\frac{6\delta_i}{r}+O(r^{-7/2}).$$

Then

$$g'(A)=c\ln\left(1+\sum_{i\in I}\left[\frac{3}{r^2}+\frac{7}{r^3}-\frac{6\delta_i}{r}+O(r^{-7/2})\right]\right)=$$

$$=c\ln\left(1+\frac{3|I|}{r^2}+\frac{7|I|}{r^3}-\sum_{i\in I}\frac{6\delta_i}{r}+O(|I|r^{-7/2})\right)=$$

$$=|\text{since }\sum_{i\in I}\frac{\delta_i}{r}\leqslant|I|r^{-11/4}|=$$

$$=c\left(\frac{3|I|}{r^2}+\frac{7|I|}{r^3}-\sum_{i\in I}\frac{6\delta_i}{r}+O(xr^{-7/2})-\frac{9|I|^2}{2r^4}+O\left(|I|^2r^{-19/4}\right)\right)=$$

$$=|\text{since }|I|\leqslant r|=$$

$$=c\left(\frac{3|I|}{r^2}+\frac{7|I|}{r^3}-\sum_{i\in I}\frac{6\delta_i}{r}+O(r^{-5/2})-\frac{9|I|^2}{2r^4}\right).$$

Now we substitute $c=\frac{r\ln r}{3}-\frac{5}{18}\ln r-\frac{1}{3}-r^{-1/6}$:

$$g'(A)=\frac{|I|\ln r}{r}-\frac{5}{6}\frac{|I|\ln r}{r^2}-\frac{|I|}{r^2}-\frac{3|I|}{r^{2+1/6}}+\frac{7}{3}\frac{|I|\ln r}{r^2}-2\ln r\sum_{i\in I}\delta_i-\frac{3}{2}\frac{|I|^2\ln r}{r^3}+O\left(\frac{\ln r}{r^{3/2}}\right).$$

Here we use the fact that $\frac{7|I|}{r^3}-\sum_{i\in I}\frac{6\delta_i}{r}+O(r^{-5/2})-\frac{9x^2}{2r^4}=O(r^{-7/4})$.

Now we are almost ready to complete the proof. Recall the lower bound from (21) for $f'(A)$:

$$f'(A)=\sum_{i\in I}f_i(A)\geqslant\frac{|I|\ln r}{r}+\sum_{i\in I}\left[\delta_i\ln\delta_i+\ln\left(\frac{r}{r-1}\right)\delta_i-\delta_i\right].$$

Thus,

$$f'(A)-g'(A)\geqslant-\frac{3|I|\ln r}{2r^2}+\frac{|I|}{r^2}+\frac{3|I|}{r^{2+1/6}}+\frac{3|I|^2\ln r}{2r^3}+O\left(\frac{\ln r}{r^{3/2}}\right)+$$

$$+\sum_{i\in I}\left[\delta_i\ln\delta_i+\ln\left(\frac{r}{r-1}\right)\delta_i-\delta_i+2\ln r\delta_i\right].$$

The above expression is minimized when $\delta_i = (r-1)/r^2$ for all $i \in I$. Hence, finally

$$f'(A) - g'(A) \geqslant -\frac{3x \ln r}{2r^2} + \frac{|I|}{r^2} + \frac{3|I|}{r^{2+1/6}} + \frac{3|I|^2 \ln r}{2r^3} - \frac{x}{r^2} + O\left(\frac{\ln r}{r^{3/2}}\right) =$$

$$= -\frac{3|I| \ln r}{2r^2} + \frac{3|I|}{r^{2+1/6}} + \frac{3|I|^2 \ln r}{2r^3} + O\left(\frac{\ln r}{r^{3/2}}\right) =$$

$$= \frac{3|I| \ln r}{2r^2}\left(\frac{|I|}{r} - 1\right) + \frac{3|I|}{r^{2+1/6}} + O\left(\frac{\ln r}{r^{3/2}}\right). \quad (23)$$

□

3.5.4 Cases

Now we have to consider different cases corresponding to the number of bad vectors $|I|$ in the matrix A. Note that

$$f(A) - g(A) \geqslant \sum_{i \notin I}(f_i(A) - g_i(A)) + f'(A) - g'(A).$$

Recal that we want to show that the above expression is bounded from below by $b \sum_{i,j=1}^{r} \varepsilon_{ij}^2$ for some $b = b(r) > 0$. Since $\sum_{i,j=1}^{r} \varepsilon_{ij}^2 < 1$ for any matrix A, it is also sufficient to show that $f(A) - g(A) \geqslant b$ for some $b = b(r) > 0$.

1. If $|I| = 0$ then everything is already proved due to (19) and (20).
2. If $|I| \geqslant r - r^{5/6}/\ln r$ then

$$f'(A) - g'(A) \geqslant \frac{3|I| \ln r}{2r^2} \cdot \left(-\frac{1}{r^{1/6} \ln r}\right) + \frac{3|I|}{r^{2+1/6}} + O\left(\frac{\ln r}{r^{3/2}}\right) =$$

$$= \frac{3|I|}{2r^{2+1/6}} + O\left(\frac{\ln r}{r^{3/2}}\right) > \frac{3}{4r^{7/6}}.$$

   Again we are done due to (19) and (20).

3. Suppose $|I| < r - r^{5/6}/\ln r$. Then

$$f'(A) - g'(A) > -\frac{3|I| \ln r}{2r^2} + O(\ln r \cdot r^{-3/2}) \geqslant -\frac{3 \ln r}{2r} + O(\ln r \cdot r^{-3/2}) > -\frac{2 \ln r}{r}.$$

   Let J denote the set of indexes of normal rows and let T denote the index set of good rows. Then $|J| + |T| > r^{5/6}/\ln r$. If $|J| \geqslant r^{4/5}$ then (20) implies that for large enough r,

$$f(A) - g(A) \geqslant \sum_{i \in J}(f_i(A) - g_i(A)) + f'(A) - g'(A) \geqslant$$

$$\geqslant r^{4/5}\frac{1}{8}r^{-7/4} \ln r - \frac{2 \ln r}{r} \geqslant \frac{1}{9}r^{-19/20} \ln r.$$

4. It remains to consider the case when $|J| < r^{4/5}$ and $0 < |I| < r - r^{5/6}/\ln r$. Let $i \in I$ be an index of a bad row. Then due to (17)

$$\sum_{i' \in T} \varepsilon_{i'i} = -\varepsilon_{ii} - \sum_{h' \in J \cup I; h' \neq i} -\varepsilon_{h'i} \leqslant$$

($\varepsilon_{ii} \geqslant \frac{1}{r} - \frac{1}{r^2} - r^{-7/4}$ and any other element is at least $-r^{-2}$)

$$\leqslant -\frac{1}{r} + \frac{1}{r^2} + r^{-7/4} + \frac{1}{r^2}\left(|J| + |I|\right) \leqslant$$

(since $|J| < r^{4/5}$ and $|I| < r - r^{5/6}/\ln r$)

$$\leqslant -\frac{1}{r} + \frac{1}{r^2} + r^{-7/4} + \frac{1}{r^2}\left(r^{4/5} + r - r^{1-1/6}/\ln r\right) = -\frac{r^{-7/6}}{\ln r}(1 + o(1)) = y.$$

So, the sum over all negative numbers is less than y:

$$\sum_{i' \in T : \varepsilon_{i'i} < 0} \varepsilon_{i'i} \leqslant y.$$

By Cauchy–Schwarz inequality

$$\sum_{i' \in T : \varepsilon_{i'i} < 0} \varepsilon_{i'i}^2 \geqslant \frac{1}{r} y^2.$$

Finally,

$$\begin{aligned}
f(A) - g(A) &\geqslant \sum_{i \in T} (f_i(A) - g_i(A)) + f'(A) - g'(A) \geqslant \\
&\geqslant \sum_{i \in T} \left(\frac{r^2}{4} \sum_{j : \varepsilon_{ij} < 0} \varepsilon_{ij}^2 \right) - \frac{3|I| \ln r}{2r^2} + O\left(\frac{\ln r}{r^{3/2}}\right) \geqslant \\
&\geqslant \frac{r^2}{4} \sum_{j \in I} \sum_{i \in T : \varepsilon_{ij} < 0} \varepsilon_{ij}^2 - \frac{3|I| \ln r}{2r^2} + O\left(\frac{\ln r}{r^{3/2}}\right) \geqslant \\
&\geqslant \frac{r^2}{4} \cdot |I| \cdot \frac{1}{r} y^2 - \frac{3|I| \ln r}{2r^2} + O\left(\frac{\ln r}{r^{3/2}}\right) = \\
&= \frac{|I|}{4r^{4/3}(\ln r)^2}(1 + o(1)) - \frac{3|I| \ln r}{2r^2} + O\left(\frac{\ln r}{r^{3/2}}\right).
\end{aligned}$$

Since $|I| \geqslant 1$, the above expression is at least $\frac{1}{4r^{4/3}(\ln r)^2}(1 + o(1))$. This completes the proof of Lemma 1.

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