Extremal Problems for Families of Subsets

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Let $[n] = \{1, 2, \ldots, n\}$ be our underlying set and let $\binom{[n]}{k}$ denote the family of its k-element subsets. We will consider families $\mathcal{F} \subset 2^{[n]}$ of subsets of $[n]$. If all of these subsets have sizes k then we call it a *k-uniform* family, otherwise it is *non-uniform*. The main goal of our investigations is to find the maximum of $|\mathcal{F}|$ under some conditions supposed on $\mathcal{F}$. The conditions are mostly given in the form that certain configurations of subsets are forbidden in $\mathcal{F}$.

The first such theorem was found by Sperner in 1928 . He proved that the maximum of $|\mathcal{F}|$ is $\binom{n}{\lfloor n/2 \rfloor}$ under the condition that $|\mathcal{F}|$ contains no pair of members F, G such that $F \subset G$. (Such families will be called *Sperner families* in what follows.) The next results were found on *intersecting families*, containing no pair of members F, G such that $F \cap G = \emptyset$. Erdős, Ko and Rado (1961) made the observation that the largest size of an intersecting family is 2^{n-1}. A more difficult result in the same paper says that the maximum size of an intersecting k-uniform family is $\binom{n-1}{k-1}$, supposing $k \leq n/2$.

The family $\mathcal{F}$ is called t-intersecting if $|F \cap G| \geq t$ holds for any two members $F, G \in \mathcal{F}$. The largest t-intersecting families were determined in 1964. But the determination of the largest t-intersecting k-uniform families proved to be more difficult. This was a result of Ahlswede and Khachatrian in 1997. In the present lecture we will show more results and problems along these lines. A recently popular direction of this theory is when a small poset P is fixed and the maximally sized family is sought under the condition that the family contains no members forming P with respect to the relation $\subset$. E.g. if P has only two comparable elements, this condition means that the family has no member containing another one as a subset: the condition of the Sperner theorem.

A "book proof" of Sperner theorem was given in 1966 by Lubell, but his proof actually gave the following inequality:

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1.$$

The Sperner theorem is an immediate consequence. Let $\mathcal{F}_i$ be the subfamily of $\mathcal{F}$ consisting of its i-element members. Introducing the notation $f_i = f_i(\mathcal{F}) = |\mathcal{F}_i|$ the inequality above can be rewritten in the following form.

$$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \leq 1.$$

The vector $(f_0(\mathcal{F}), f_1(\mathcal{F}), \ldots, f_n(\mathcal{F}))$ is called the *profile vector* of $\mathcal{F}$. Take the profile vectors of all Sperner families. They form a set of points with integer coordinates in the $n+1$-dimensional Euclidean space. Take its convex hall. It is easy to see from the inequality above that the extreme points of this convex hull are the zero vector and $\big(0, 0, \ldots, \binom{n}{i}, \ldots, 0\big)$ $(0 \leq i \leq n)$.

We will survey known results for the extreme points of other classes of families and pose related open problems for largest families and extreme points.