Introduction to Extremal Set Theory

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Let $[n] = \{1, 2, \ldots, n\}$ be our underlying set and let $\binom{[n]}{k}$ denote the family of its k-element subsets. We will consider families $\mathcal{F} \subset 2^{[n]}$ of subsets of $[n]$. If all of these subsets have sizes k then we call it a k-*uniform* family, otherwise it is *non-uniform*. The main goal of our investigations is to find the maximum of $|\mathcal{F}|$ under some conditions supposed on $\mathcal{F}$. The conditions are mostly given in the form that certain configurations of subsets are forbidden in $\mathcal{F}$.

The first such theorem was found by Sperner in 1928 . He proved that the maximum of $|\mathcal{F}|$ is $\binom{n}{\lfloor n/2 \rfloor}$ under the condition that $|\mathcal{F}|$ contains no pair of members F, G such that $F \subset G$. (Such families will be called *Sperner families* in what follows.) The next results were found on *intersecting families*, containing no pair of members F, G such that $F \cap G = \emptyset$. Erdős, Ko and Rado (1961) made the observation that the largest size of an intersecting family is 2^{n-1}. A more difficult result in the same paper says that the maximum size of an intersecting k-uniform family is $\binom{n-1}{k-1}$, supposing $k \leq n/2$.

The family $\mathcal{F}$ is called t-intersecting if $|F \cap G| \geq t$ holds for any two members $F, G \in \mathcal{F}$. The largest t-intersecting families were determined in 1964. But the determination of the largest t-intersecting k-uniform families proved to be more difficult. This was a result of Ahlswede and Khachatrian in 1997. In the present lectures we will show some results, problems and methods along these lines. A recently popular direction of this theory is when a small poset P is fixed and the maximally sized family is sought under the condition that the family contains no members forming P with respect to the relation $\subset$. E.g. if P has only two comparable elements, this condition means that the family has no member containing another one as a subset: the condition of the Sperner theorem.