Matrix Theory and Applications

Mehdi Golafshan, Alexei Kanel-Belov

Spring, 2021

E-mail: mgolafshan@yandex.ru | Web: mipt.ru
Office Hours: Sat, 18-21 | Class Hours:
Office: golafshan.m.mehdi (my Skype) | Class Room: Online

Course Description

Matrix theory plays an important role in many core artificial intelligence (AI) areas, including machine learning, neural networks, support vector machines (SVMs) and evolutionary computation. This course offers a comprehensive and in-depth discussion of matrix theory and methods for these four core areas of AI, while also approaching AI from a theoretical matrix perspective.

Purpose of the Course

- Strengthening the Linear Algebra course for university students as a second course
- Teaching the concepts needed by students for research with faculty members
- Improving students' problem solving in linear algebra and their success in the IMC Student Olympiad

Course Outline

- **Elementary Linear Algebra Review:** Vector Spaces, Matrices and Determinants, Linear Transformations and Eigenvalues, Inner Pro duct Spaces
- **Partitioned Matrices, Rank, and Eigenvalues:** Elementary Operations of Partitioned Matrices, The Determinant and Inverse of Partitioned Matrices, The Rank of Pro duct and Sum, The Eigenvalues of AB and BA, The Continuity Argument and Matrix Functions, Localization of Eigenvalues: The Gerosgorin Theorem
- **Matrix Polynomials and Canonical Forms:** Commuting Matrices, Matrix Decompositions, Annihilating Polynomials of Matrices, Jordan Canonical Forms, The matrices associated with a matrix A
- **Numerical Ranges, Matrix Norms, and Special Operations:** Numerical Range and Radius, Matrix Norms, The Kronecker and Hadamard Products, Compound Matrices

- **Special Types of Matrices:** Idempotence, Nilpotence, Involution, and Projections, Tridiagonal Matrices, Circulant Matrices, Vandermonde Matrices, Hadamard Matrices, Permutation and Doubly Stochastic Matrices, Nonnegative Matrices
- **Unitary Matrices and Contractions:** Properties of Unitary Matrices, Real Orthogonal Matrices, Metric Space and Contractions, Contractions and Unitary Matrices, The Unitary Similarity of Real Matrices, A Trace Inequality of Unitary Matrices
- **Positive Semidefinite Matrices:** Positive Semidefinite Matrices, A Pair of Positive Semidefinite Matrices, Partitioned Positive Semidefinite Matrices, Schur Complements and Determinant Inequalities, The Kronecker and Hadamard Pro ducts of Positive Semidefinite Matrices, Schur Complements and the Hadamard Product, The Wielandt and Kantorovich Inequalities
- **Hermitian Matrices:** Hermitian Matrices and Their Inertias, The Product of Hermitian Matrices, The Min-Max Theorem and Interlacing Theorem, Eigenvalue and Singular Value Inequalities, Eigenvalues of Hermitian Matrices, A Triangle Inequality for the Matrix $(A^*A)^{1/2}$
- **Normal Matrices:** Equivalent Conditions, Normal Matrices with Zero and One Entries, Normality and Cauchy–Schwarz–Type Inequalities, Normal Matrix Perturbation
- **Majorization and Matrix Inequalities:** Basic Properties of Majorization, Majorization and Stochastic Matrices, Majorization and Convex Functions, Majorization of Diagonal Entries, Eigenvalues, and Singular Values, Majorization for Matrix Product, Majorization and Unitarily Invariant Norms

Summary

Elementary Linear Algebra Review

We briefly review, mostly without proof, the basic concepts and results taught in an elementary linear algebra course. The subjects are vector spaces, basis and dimension, linear transformations and their eigenvalues, and inner product spaces.

Partitioned Matrices, Rank, and Eigenvalues

We begin with the elementary operations on partitioned (block) matrices, followed by discussions of the inverse and rank of the sum and product of matrices. We then present four different proofs of the theorem that the products AB and BA of matrices A and B of sizes $m \times n$ and $n \times m$, respectively, have the same nonzero eigenvalues. At the end of this chapter we discuss the often-used matrix technique of continuity argument and the tool for localizing eigenvalues by means of the Gerosgorin discs.

Matrix Polynomials and Canonical Forms

This part is devoted to matrix decompositions. The main studies are on the Schur decomposition, spectral decomposition, singular value decomposition, Jordan decomposition, and numerical range. Attention is also paid to the polynomials that annihilate matrices, especially the minimal and characteristic polynomials, and to the similarity of a complex matrix to a real matrix. At the end we introduce three important matrix operations: the Hadamard product, the Kronecker product, and compound matrices.

Numerical Ranges, Matrix Norms, and Special Operations

This part is devoted to a few basic topics on matrices. We first study the numerical range and radius of a square matrix and matrix norms. We then introduce three important special matrix operations: the Kronecker product, the Hadamard product, and compound matrices.

Special Types of Matrices

This part studies special types of matrices. They are: idempotent matrices, nilpotent matrices, involutary matrices, projection matrices, tridiagonal matrices, circulant matrices, Vandermonde matrices, Hadamard matrices, permutation matrices, doubly stochastic matrices, and nonnegative matrices. These matrices are often used in many subjects of mathematics and in other fields.

Unitary Matrices and Contractions

This part studies unitary matrices and contractions. First section gives basic properties of unitary matrices, second section discusses the structure of real orthogonal matrices under similarity, and third section develops metric spaces and the fixed-point theorem of strict contractions. Fourth deals with the connections of contractions with unitary matrices, fifth section concerns the unitary similarity of real matrices, and the last part presents a trace inequality for unitary matrices, relating the average of the eigenvalues of each of two unitary matrices to that of their product.

Positive Semidefinite Matrices

This part studies the positive semidefinite matrices, concentrating primarily on the inequalities of this type of matrix. The main goal is to present the fundamental results and show some often used techniques. In the first part gives the basic properties, second part treat the Lowner partial ordering of positive semidefinite matrices, and third part presents some inequalities of principal submatrices. Fourth part derives inequalities of partitioned positive semidefinite matrices using Schur complements, and parth five and six investigate the Hadamard product of the positive semidefinite matrices. Finally, part seven shows the Cauchy–Schwarz type matrix inequalities and the Wielandt and Kantorovich inequalities.

Hermitian Matrices

This part contains fundamental results of Hermitian matrices and demonstrates the basic techniques used to derive the results. First part presents equivalent conditions to matrix Hermitity, second part gives some trace inequalities and discusses a necessary and sufficient condition for a square matrix to be a product of two Hermitian matrices, and third part develops the min-max theorem and the interlacing theorem for eigenvalues. Fourth part deals with the eigenvalue and singular value inequalities for the sum of Hermitian matrices, and part five shows a matrix triangle inequality.

Normal Matrices

A great deal of elegant work has been done for normal matrices. The goal of this part is to present basic results and methods on normal matrices. Part one gives conditions equivalent to the normality of matrices, second part focuses on a special type of normal matrix with entries consisting of zeros and ones, part three studies the positive semidefinite matrix $(A^*A)^{1/2}$ associated with a matrix A, and finally part four compares two normal matrices.

Majorization and Matrix Inequalities

Majorization is an important tool in deriving matrix inequalities of eigenvalues, singular values, and matrix norms. In this part we introduce the concept of majorization, present its basic properties, and show a variety of matrix inequalities in majorization.

Prerequisites/Corequisites

Linear Algebra and Geometry (first course in linear algebra), Mathematical Analysis (firs course in analysis)

Main References

[1] Carl D. Meyer, Matrix analysis and applied linear algebra,

[2] F. Zhang , Matrix Theory Basic Results and Techniques, Springer, New York, 2011.

[3] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.

[4] Denis Serre, Matrices: Theory and Applications, Springer-Verlag New York, 2010.

[5] V. V. Prasolov, Problems and Theorems in Linear Algebra, American Mathematical Society, Providence, RI, 1994.

Grading Policy

The grade will count the assessments using the following proportions:

- **100%** of your grade will be determined by 10 series hometasks (10% each).

Hometasks:

A series of exercises is assigned to each section.